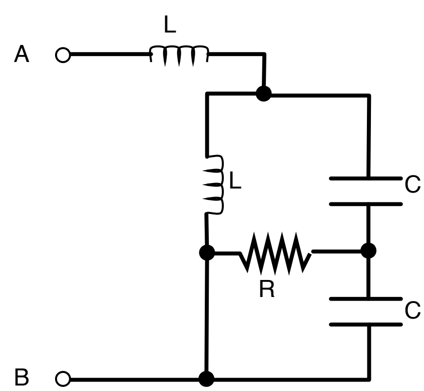
MAE 140 - Linear Circuits - Fall 10
Final, December 8

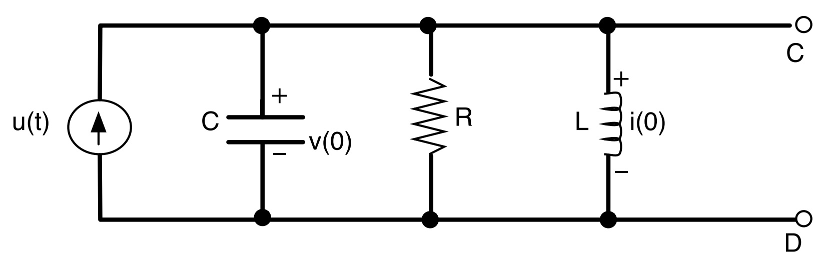
Instructions:

1. This exam is open book: you may use whatever written materials you choose, including your class notes and textbook
2. You may use a hand calculator with no communication capabilities
3. You have 170 minutes
4. Write your name, student number, and instructor on your blue book

1. **Equivalent Circuits**



(a) Circuit for part (a)

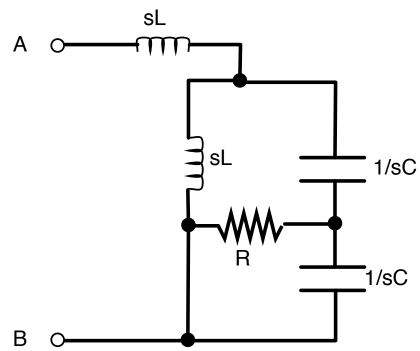


(b) Circuit for part (b)

Figure 1: Circuits for Problem 1

(a) [4 points] Assuming zero initial conditions, find the impedance equivalent to the circuit in Figure 1(a) as seen from the terminals A and B. The answer should be given as the ratio of two polynomials.

Solution:
With zero initial conditions, the circuit in the s-domain is



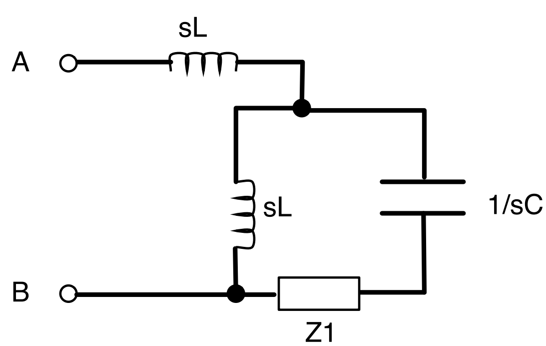
(+1 point)

We then start by lumping together impedances from port B to port A as follows. First, note that R and C are connected in parallel. This gives:

$$Z_1 = \frac{\frac{R}{sC}}{R + \frac{1}{sC}} = \frac{R}{sCR + 1}.$$

(+1 point)

Now we have Z_1 and C connected in series:

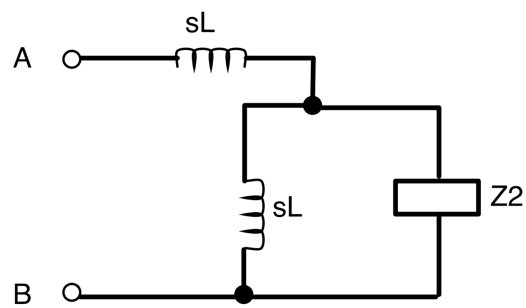


We can sum these together and obtain:

$$Z_2 = Z_1 + \frac{1}{sC} = \frac{R}{sCR + 1} + \frac{1}{sC} = \frac{2sRC + 1}{(sCR + 1)sC}.$$

(+1 point)

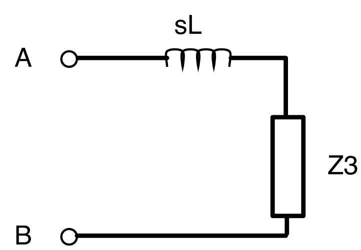
After doing this, we observe that the inductor sL and Z_2 have the same voltage accross terminals:



Thus, they are connected in parallel and we can obtain an equivalent impedance:

$$
\begin{aligned}
Z_3 = \frac{Z_2 sL}{sL + Z_2} &= \frac{(2sRC+1)sL}{(sCR+1)sC} \times \frac{(sCR+1)sC}{s^2LC(sCR+1)+2sRC+1} \\
&= \frac{(2sRC+1)sL}{s^2LC(sCR+1)+2sRC+1} \\
&= \frac{(2sRC+1)sL}{s^3LC^2R+s^2LC+2sRC+1} \qquad \text{(+1 point)}
\end{aligned}
$$

Finally, the new Z_3 is connected in series with the second inductor:



This leads to:

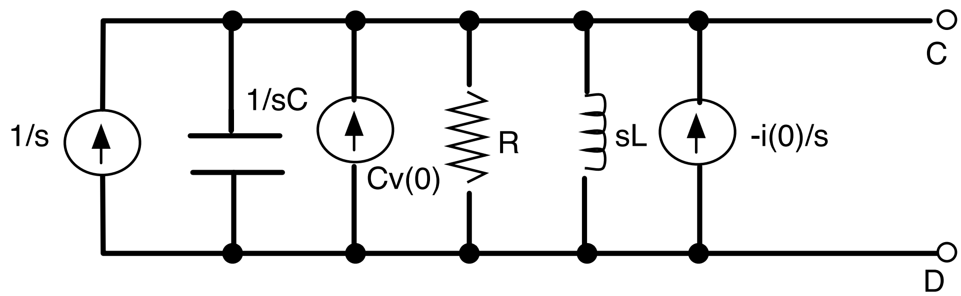
$$
\begin{aligned}
Z_{\text{eq}} &= sL + Z_3 \\
&= sL + \frac{(2sRC+1)sL}{s^3LC^2R+s^2LC+2sRC+1} \\
&= \frac{s^4L^2C^2R + s^3L^2C + 2s^2LRC + sL + 2s^2LRC + sL}{s^3LC^2R+s^2LC+2sRC+1} \\
&= \frac{s^4L^2C^2R + s^3L^2C + 4s^2LRC + 2sL}{s^3LC^2R+s^2LC+2sRC+1} \qquad \text{(+1 point)}
\end{aligned}
$$

(NOTE TO GRADERS: Students should get 1 point per item based on their reasoning. If they make an algebra mistake discount 1 point from the total points.)

(b) [6 points] Assuming that the initial conditions of the inductor and capacitor are as indicated in the diagram, redraw the circuit shown in Figure 1(b) in the s-domain. Then use source transformations to find the s-domain Thevenin equivalent of this circuit as seen from the terminals C and D. The Thevenin impedance and voltage should be given as the ratio of two polynomials.

Hint: Use an equivalent model for the inductor and the capacitor where the initial conditions appear as current sources.

Solution:
Following the hint, we use current sources to take into account the initial conditions of the capacitor and inductor. The transformed circuit in the s domain is



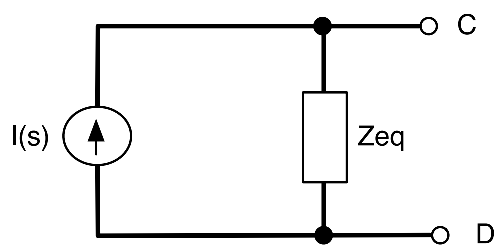
(+1 point)

Since the sources and impedances are in parallel, this circuit is equivalent to another with one current source

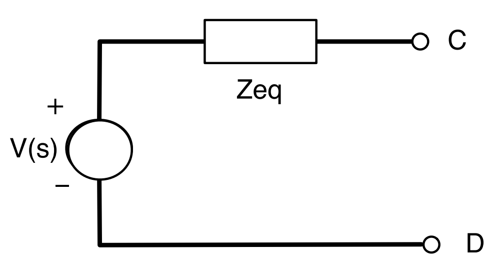
$$I(s) = \frac{1}{s} + Cv(0) - \frac{i(0)}{s} = \frac{sCv(0) + 1 - i(0)}{s}, \qquad \text{(+1 point)}$$

and impedance

$$Z_{\text{eq}} = \frac{1}{sC + \frac{1}{R} + \frac{1}{sL}} = \frac{sLR}{s^2CRL + sL + R} \qquad \text{(+1 point)}$$



This is the Norton equivalent of the circuit in the s domain between terminals C and D. Now we use an equivalent source transformation to obtain the Thevenin equivalent



as follows:

$$V_{\text{TH}}(s) = \frac{sCv(0) + 1 - i(0)}{s} \times \frac{sLR}{s^2CRL + sL + R} = \frac{LR(1 - i(0)) + sLRCv(0)}{s^2CRL + sL + R} \quad \text{(+1 point)}$$

and

$$Z_{\text{TH}} = Z_{\text{eq}} = \frac{sLR}{s^2CRL + sL + R}. \quad \text{(+1 point)}$$

(NOTE TO GRADERS: Students should get 1 point per item based on their reasoning. If they make an algebra mistake discount 1 point from the total points.)

2. **Laplace Domain Circuit Analysis**

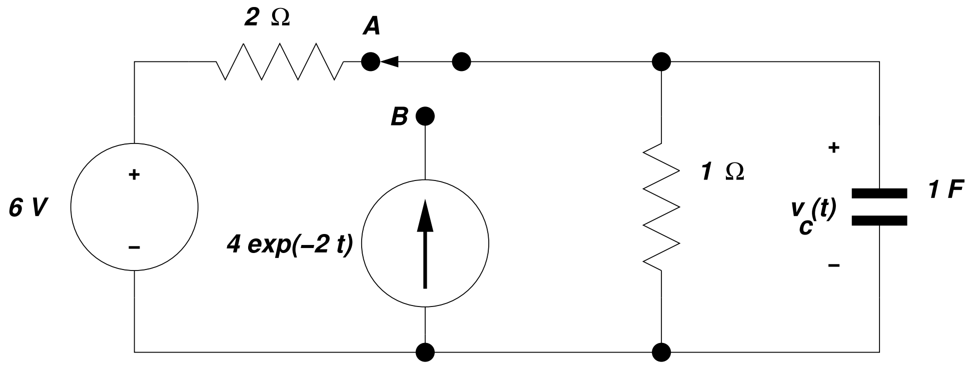


Figure 2: Circuit for Laplace domain analysis.

Consider the circuit in Figure 2 where the voltage source is constant and equal to $6V$. The switch has been kept in position A for a very long time. At $t = 0$ it is moved to position B.

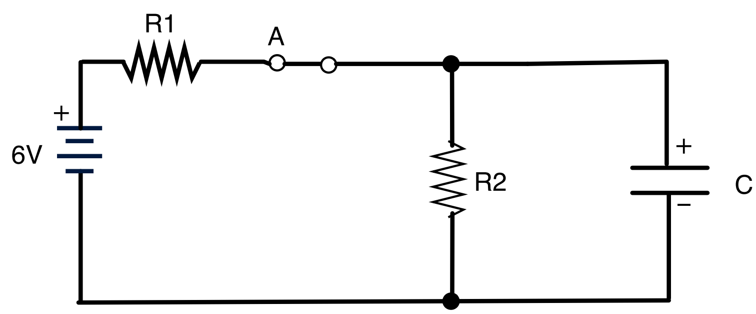
(a) [2 points] Show that the initial capacitor voltage is given by:

$$v_C(0^-) = 2\,V.$$

[Show your work]

Solution:

Initially the circuit is connected to a constant voltage source as in the figure below:



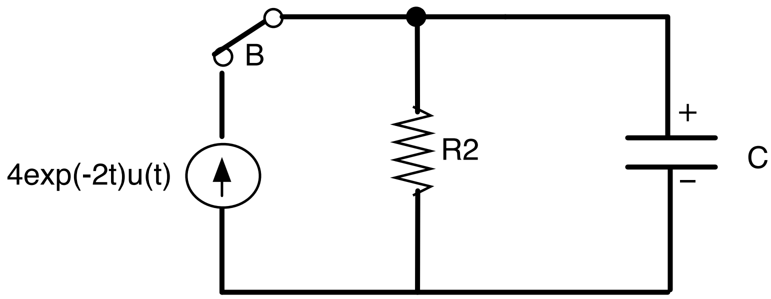
Over a long time the source will charge the capacitor to a constant voltage when it will behave as an open circuit. (+1 point)

The voltage across the capacitor will be equal to the voltage across the 1Ω resistor. By applying voltage division, we find that:

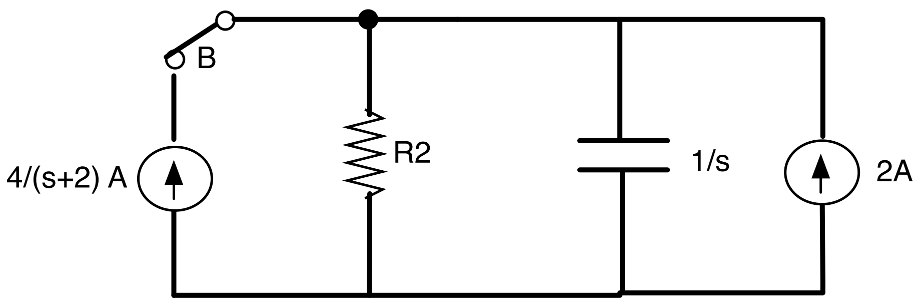
$$v_c(0^-) = \frac{1}{1+2}6V = \frac{6}{3}V = 2V \qquad \text{(+1 point)}$$

(b) [2 points] Use the above initial condition to transform the circuit into the s-domain for $t \geq 0$. Use an equivalent model for the capacitor where the initial condition appears as a current source.
[Show your work]

Solution:
At time $t = 0$, the switch is at position B and the circuit becomes:



This circuit is transformed into the s domain to another one as follows (we have used that $C = 1F$, and that $R_2 = 1\Omega$)



(+1 point)

Where the current sources have values

$$I_1(s) = \frac{4}{s+2}, \qquad I_2(s) = 2.$$ (+1 point)

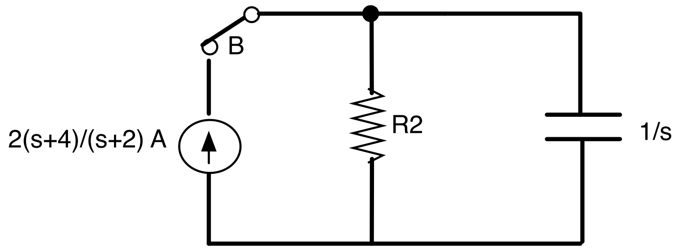
(c) [6 points] Use s-domain circuit analysis and inverse Laplace transforms to show that the voltage accross the capacitor is

$$v_C(t) = [6\exp(-t) - 4\exp(-2t)]u(t).$$

Solution:
To find the voltage across the capacitor, we first sum the current sources to obtain

$$I(s) = \frac{4}{s+2} + 2 = \frac{4+4+2s}{s+2} = \frac{2s+8}{s+2}$$ (+1 point)

This simplifies the circuit to



Using current division, we can find the current going through the capacitor as follows:

$$I_C(s) = \frac{s}{s+1}\frac{2(s+4)}{s+2} = \frac{2s(s+4)}{(s+1)(s+2)} \qquad \text{(+1 point)}$$

Now, by Ohm's law, we have:

$$V_C(s) = I_C(s)\frac{1}{s} = \frac{2(s+4)}{(s+1)(s+2)} \qquad \text{(+1 point)}$$

To find the voltage across the capacitor in the s domain, we take the inverse Laplace transform of $V_C(s)$. First, we decompose it into simple fractions:

$$V_C(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

The values of the constants can be obtained applying the residue formulas:

$$A = \lim_{s\to-1} \frac{2(s+4)}{s+2} = \frac{2(-1+4)}{(-1+2)} = 6,$$
$$B = \lim_{s\to-2} \frac{2(s+4)}{s+1} = \frac{2(-2+4)}{-2+1} = -4$$

In other words,

$$V_C(s) = \frac{6}{s+1} - \frac{4}{s+2} \qquad \text{(+2 points)}$$

The inverse Laplace transform becomes:

$$v_C(t) = (6\mathrm{e}^{-t} - 4\mathrm{e}^{-2t})u(t) \qquad \text{(+1 point)}$$

(d) [1 point (bonus)] Identify the forced and natural components of $v_C(t)$.

Solution:
The natural component of the voltage is the one that comes from the initial condition $v_c(0^-)$ while the forced component of the voltage is the one produced by the voltage source $4\exp(-2t)$.

In this way:

$$\begin{aligned} &\text{Natural component:} \quad 6\exp(-t)u(t), \\ &\text{Forced component:} \quad -4\exp(-2t)u(t). \end{aligned}$$

(+1 bonus point)

3. **Nodal and Mesh Analysis**

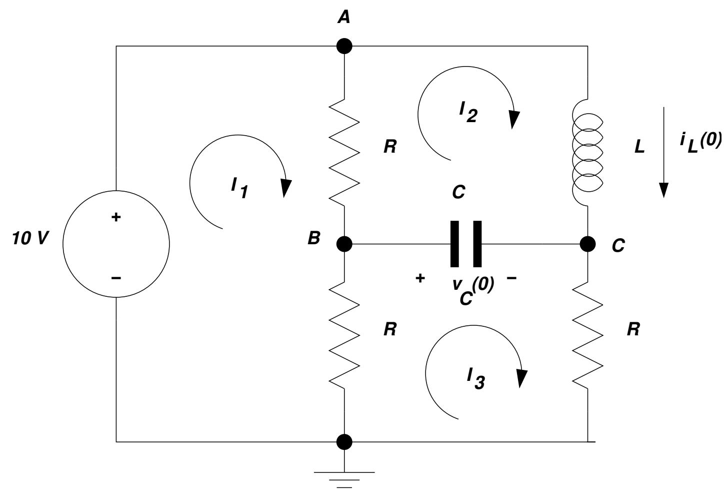
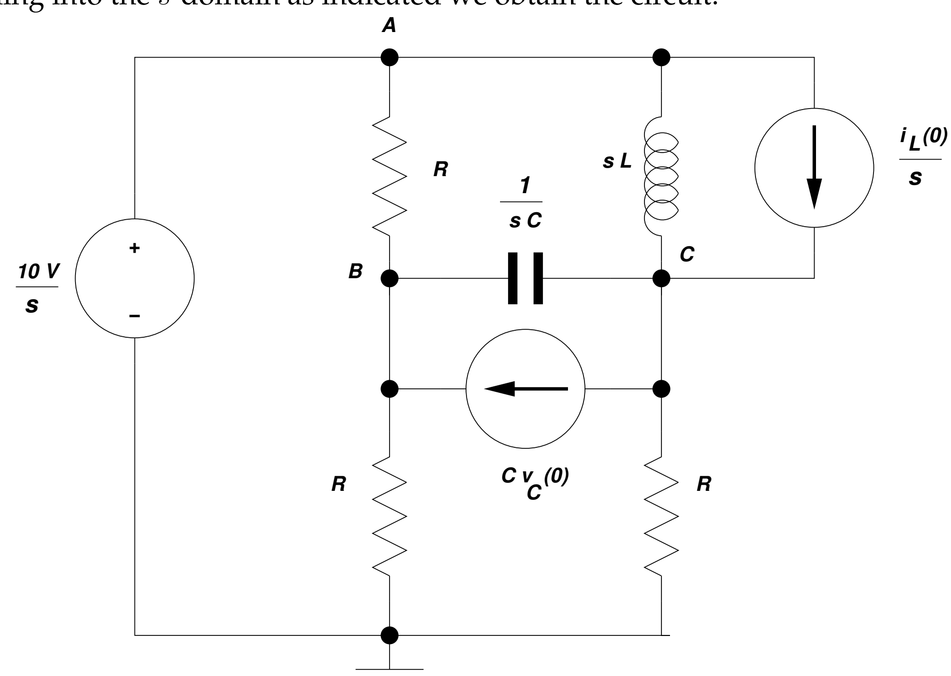


Figure 3: Nodal and mesh analysis circuit.

In your answer to this problem, clearly indicate the voltages, currents and the equations that need to be solved in each part.

(a) [5 points] Formulate node-voltage equations in the s-domain for the circuit in Figure 3. Use the reference node and other labels as shown in the figure. Use the initial conditions indicated in the figure! Transform initial conditions on the capacitor and on the inductor into current sources.

Solution:
Transforming into the s-domain as indicated we obtain the circuit:



(+1 point)

Here we can use method #2 to eliminate the voltage source and determine the voltage at node A:

$$V_A(s) = \frac{10}{s} \qquad \text{(+1 point)}$$

Then we can write the node-voltage equations by inspection at all other nodes

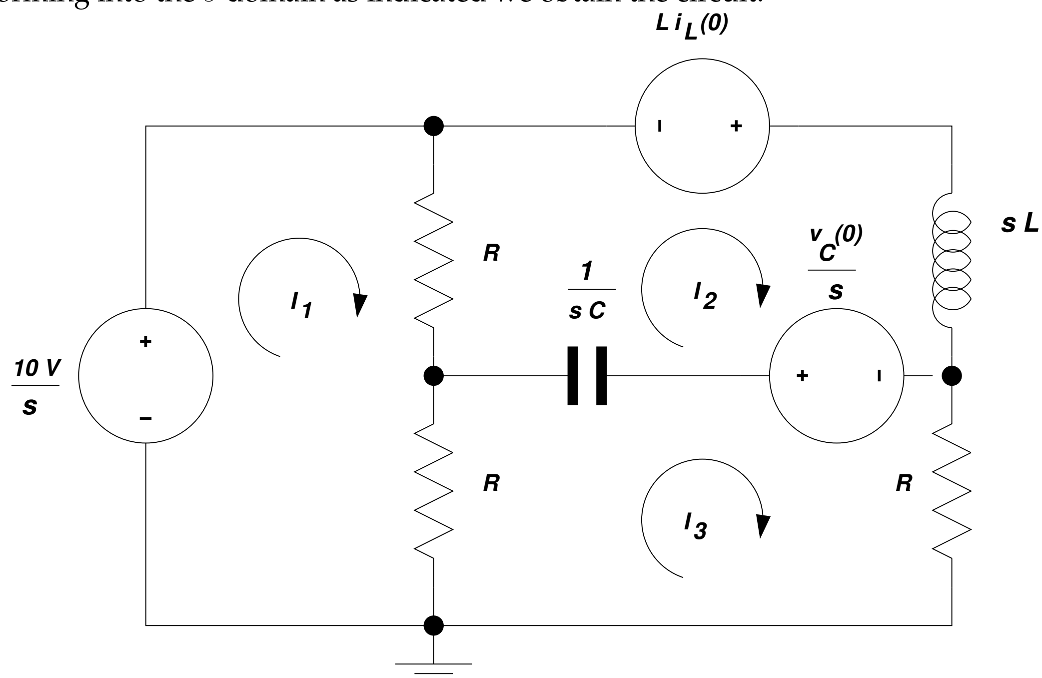
$$\begin{bmatrix} -\frac{1}{R} & \frac{2}{R} + sC & -sC \\ -\frac{1}{sL} & -sC & R + sC + \frac{1}{sL} \end{bmatrix} \begin{pmatrix} V_A(s) \\ V_B(s) \\ V_C(s) \end{pmatrix} = \begin{pmatrix} Cv_C(0) \\ \frac{i_L(0)}{s} - Cv_c(0) \end{pmatrix} \qquad \text{(+2 points)}$$

The above three equations must be solved simultaneously for the node-voltages $V_A(s)$, $V_B(s)$ and $V_C(s)$. (+1 point)

(b) [5 points] Formulate mesh-current equations in the s-domain for the circuit in Figure 3. Use the mesh currents shown in the figure. Use the initial conditions indicated in the figure! Transform initial conditions on the capacitor and on the inductor into voltage sources.

Solution:

Transforming into the s-domain as indicated we obtain the circuit:

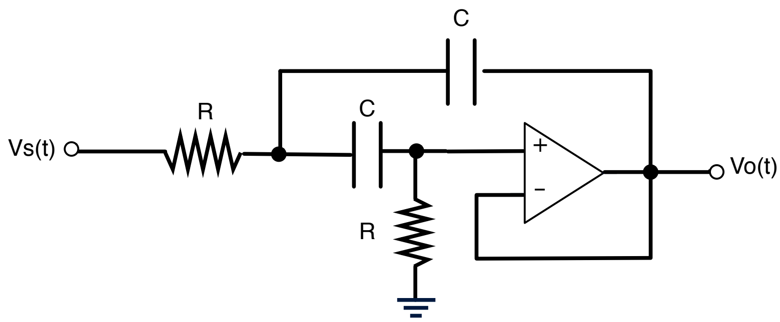


(+1 point)

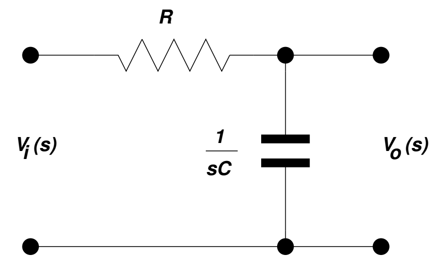
This time all three sources are voltage sources and no transformations are needed. Writing the mesh-current equations by inspection we obtain

$$\begin{bmatrix} 2R & -R & -R \\ -R & R+\dfrac{1}{sC}+sL & -\dfrac{1}{sC} \\ -R & -\dfrac{1}{sC} & 2R+\dfrac{1}{sC} \end{bmatrix} \begin{pmatrix} I_1(s) \\ I_2(s) \\ I_3(s) \end{pmatrix} = \begin{pmatrix} \dfrac{10}{s} \\ \dfrac{v_C(0)}{s} - Li_L(0) \\ -\dfrac{v_C(0)}{s} \end{pmatrix} \qquad \text{(+3 points)}$$

The above three equations must be solved simultaneously for the mesh-currents $I_1(s)$, $I_2(s)$ and $I_3(s)$. (+1 point)



(a) Circuit for frequency response analysis



(b) Circuit for filter design

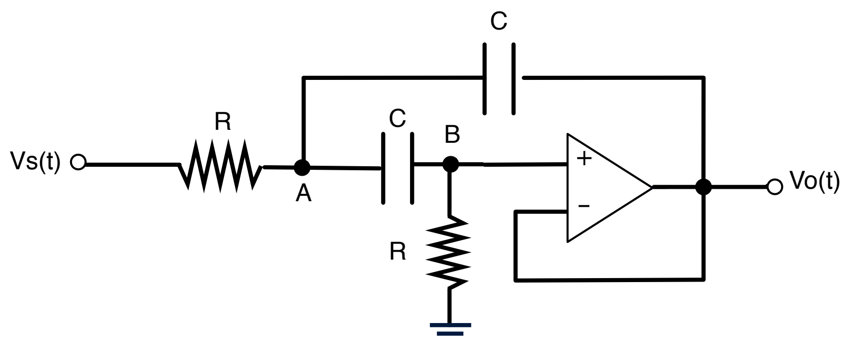
4. **Frequency Response Analysis**

   (a) [3 points] Consider the circuit in Figure 4(a). Assuming zero initial conditions, show that the transfer function from $V_s(s)$ to $V_o(s)$ is given by:

$$T(s) = \frac{sRC}{1 + 3sRC}$$

[Show your work]

Solution:
We will use nodal analysis to find the transfer function $T(s)$. Following the s-domain diagram



we first obtain two node equations for node A and node B:

$$\text{Node A:} \qquad (V_A - V_s)\frac{1}{R} + (V_A - V_B)Cs + (V_A - V_o)Cs = 0\,,$$

$$\text{Node B:} \qquad (V_B - V_A)Cs + V_B\frac{1}{R} = 0\,,$$

which are equivalent to:

$$\text{Node A:} \qquad V_A\left(\frac{1}{R} + 2Cs\right) - V_BCs - V_oCs = V_s\frac{1}{R}\,,$$

$$\text{Node B:} \qquad V_B\left(\frac{1}{R} + Cs\right) - V_ACs = 0\,.$$

(+1 point)

Then we use the Op-amp constraint equations to find that:

$$V_B = V_P, \qquad V_P = V_N, \qquad V_N = V_o, \qquad \Rightarrow V_B = V_o\,. \qquad \text{(+1 point)}$$

On the other hand, from Node B equation, we obtain that:

$$V_A = V_o\left(1 + \frac{1}{CRs}\right)$$

Substituting this expression into Node A equation leads to:

$$V_s\frac{1}{R} = V_o\left[\left(1 + \frac{1}{CRs}\right)\left(\frac{1}{R} + 2Cs\right) - 2Cs\right]$$

From here,

$$\begin{aligned} V_s &= V_oR\left[\frac{1}{R} + 2Cs + \frac{1}{CR^2s} + \frac{2}{R} - 2Cs\right] \\ &= V_o\left[3 + \frac{1}{RCs}\right] = V_o\left[\frac{3RCs+1}{RCs}\right] \end{aligned}$$

This implies that:

$$V_o(s) = T(s)V_s(s) = \frac{RCs}{1+3RCs}V_s(s). \qquad \text{(+1 point)}$$

(b) [4 points] Suppose now that $R = 1\Omega$, $C = 1F$. Compute the magnitude and phase of $T(s)$ at $j\omega$ where $\omega \in \{0, +\infty, \frac{\pi}{3}, \frac{\pi}{6}\}$. Use these values to sketch the magnitude of the frequency response of the circuit. Is the circuit a low-pass, high-pass or band-pass filter?
[Explain your answer]

Solution:
If $R = 1\Omega$ and $C = 1F$, then the transfer function expression simplies to:

$$T(s) = \frac{s}{1+3s}.$$

From here, the frequency response is the complex function:

$$T(j\omega) = \frac{j\omega}{1+3j\omega}, \qquad \omega \geq 0.$$

Its magnitude becomes:

$$|T(j\omega)| = \frac{|j\omega|}{|1+3j\omega|} = \frac{\omega}{\sqrt{1+9\omega^2}}, \qquad \omega \geq 0.$$

And its phase is given by:

$$\angle T(j\omega) = \angle j\omega - \angle(1+3j\omega) = \arctan\left(\frac{\omega}{0}\right) - \arctan\left(\frac{3\omega}{1}\right)$$

Substituting the values for ω provided in the problem statement, we obtain the following:

$$\omega = 0, \qquad |T(0)| = \frac{0}{\sqrt{1+9\times 0^2}} = 0, \qquad \angle T(0) = \arctan(\frac{0}{0}) - \arctan(0) = 0$$

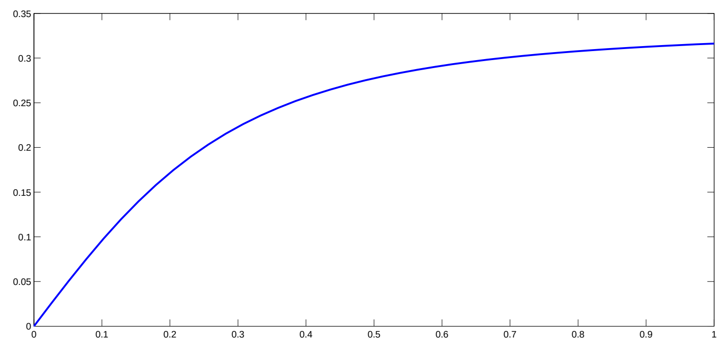
$$\omega = +\infty \qquad |T(j\infty)| = \frac{\infty}{\sqrt{1+9\times\infty^2}} = \frac{1}{3}, \qquad \angle T(j\infty) = \frac{\pi}{2} - \arctan(+3\infty) = \frac{\pi}{2} - \frac{\pi}{2} = 0\,.$$

$$\omega = \frac{1}{6} \qquad |T(j\frac{1}{6})| = \frac{\frac{1}{6}}{\sqrt{1+9\frac{1}{36}}} = \frac{1}{3\sqrt{5}}, \qquad \angle T(j\frac{1}{6}) = \frac{\pi}{2} - \arctan(\frac{1}{2})\,.$$

$$\omega = \frac{1}{3} \qquad |T(j\frac{1}{3})| = \frac{\frac{1}{3}}{\sqrt{1+9\frac{1}{9}}} = \frac{1}{3\sqrt{2}}, \qquad \angle T(j\frac{1}{3}) = \frac{\pi}{2} - \arctan(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\,.$$

(+3 points)

A graph of the magnitude



shows that the magnitude graph increases from zero $|T(0)| = 0$ to the highest value $|T(\infty)| = \frac{1}{3}$, indicating that this is a *high-pass filter*. In other words, low frequencies will be removed by the filter while high frequencies are passed with an attenuation of $\frac{1}{3}$. (+1 point)

(c) [2 points] Using the same values for R and C as in part (b), compute the steady state response of an input voltage $v(t) = 3\cos(\frac{\pi}{6}t + \frac{\pi}{4})$.

Solution:
Finally, to compute the steady-state response of an input $v(t) = 3\cos(\frac{t}{6} + \frac{\pi}{4})$, we use the frequency response values for $\omega = \frac{1}{6}$. In this way:

$$v_{SS}(t) = 3|T(j\frac{1}{6})|\cos(\frac{t}{6} + \angle T(j\frac{1}{6}) + \frac{\pi}{4}) = \frac{1}{\sqrt{5}}\cos(\frac{t}{6} + \frac{3\pi}{4} - \arctan(\frac{1}{2}))\,.$$

(+1 point for correct idea)

(+1 point for correct values)

5. **Filter Design**

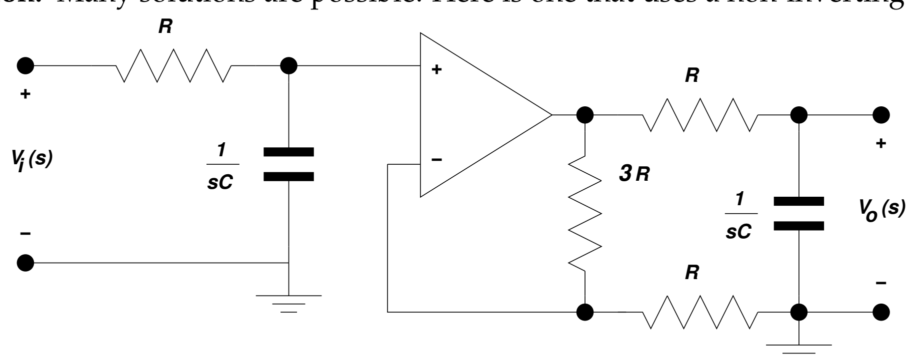
(a) [2 points] Show that the circuit in Figure 4(b), already shown in the s-domain, is a filter with transfer function

$$T_1(s) = \frac{V_o(s)}{V_i(s)} = \frac{\omega_c}{s+\omega_c}, \qquad\qquad \omega_c = \frac{1}{RC}.$$

> **Solution:** The circuit is a voltage divider, hence:
>
> $$V_o(s) = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} V_i(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} V_i(s)$$
>
> from where $T_1(s) = V_o(s)/V_i(s)$. (+2 points)

(b) [5 points] Use no more than one Op-Amp to combine two copies of the circuit in Figure 4(b) in order to obtain a new circuit with transfer function

$$T_2(s) = \pm \frac{4\,\omega_c^2}{(s+\omega_c)^2}.$$

> **Solution:** Many solutions are possible. Here is one that uses a non-inverting Op-Amp block:
>
> 
>
>
> (+2 points)
>
> The input to the non-inverting Op-Amp stage is
>
> $$V_P(s) = \frac{\omega_c}{s+\omega_c} V_i(s) \qquad \text{(+1 point)}$$
>
> The output of the Op-Amp stage is
>
> $$V_o'(s) = \frac{R+3R}{R} V_p(s) = 4V_p(s) \qquad \text{(+1 point)}$$
>
> Taking $V_o'(s)$ as the input of the filter yields
>
> $$V_o(s) = \frac{\omega_c}{s+\omega_c} V_o'(s) = 4\frac{\omega_c}{s+\omega_c} V_p(s) = 4\frac{\omega_c^2}{(s+\omega_c)2} V_i(s) \qquad \text{(+1 point)}$$

(c) [3 points] Compute the gain and the cutoff frequency of the filter you have just designed.

(**Hint:** Recall that gain is the maximum value attained by the magnitude ot the frequency response of a filter and that the cutoff frequency is the frequency at which the gain of a filter is reduced by a factor of $1/\sqrt{2}$.)

Solution: The maximum value of $|T_2(j\omega)|$ is at $\omega = 0$ because the maximum of $T_1(j\omega)$ is at $\omega = 0$. Hence the gain is

$$K = \max_{\omega} |T_2(j\omega)| = |T_2(j0)| = 4. \qquad \text{(+1 point)}$$

In order to compute the cutoff frequency we need to compute the magnitude of the frequency response:

$$|T_2(j\omega)| = \frac{4\,\omega_c^2}{|(j\omega + \omega_c)^2|} = \frac{4\,\omega_c^2}{|2j\,\omega_c\,\omega + \omega_c^2 - \omega^2|} = \frac{4\,\omega_c^2}{\sqrt{4\,\omega_c^2\,\omega^2 + (\omega_c^2 - \omega^2)^2}}. \qquad \text{(+1 point)}$$

The cutoff frequency $\bar{\omega}$ is the frequency at which

$$\frac{K}{\sqrt{2}} = \frac{4}{\sqrt{2}} = |T_2(j\bar{\omega})|$$

That is at which

$$\sqrt{4\,\omega_c^2\,\bar{\omega}^2 + (\omega_c^2 - \bar{\omega}^2)^2} = \sqrt{2}\,\omega_c^2. \qquad \text{(+1 point)}$$

Squaring both terms we get

$$4\,\omega_c^2\,\bar{\omega}^2 + (\omega_c^2 - \bar{\omega}^2)^2 = 2\,\omega_c^4$$

or

$$0 = \bar{\omega}^4 + \omega_c^4 - 2\,\omega_c^2\,\bar{\omega}^2 + 4\,\omega_c^2\,\bar{\omega}^2 - 2\,\omega_c^4 = \bar{\omega}^4 + 2\,\omega_c^2\,\bar{\omega}^2 - \omega_c^4.$$

This is a second order equation in $\bar{\omega}^2$ with roots

$$\bar{\omega}^2 = \frac{-2\omega_c^2 \pm \sqrt{4\omega_c^4 + 4\omega_c^4}}{2} = (-1 \pm \sqrt{2})\omega_c^2$$

the only positive root being

$$\bar{\omega}^2 = (\sqrt{2} - 1)\omega_c^2$$

from where

$$\bar{\omega} = \omega_c\sqrt{\sqrt{2} - 1} \approx 0.64\,\omega_c. \qquad \text{(+1 bonus point)}$$

(d) [1 point (bonus)] What type of filter have you designed?

Solution:
A low-pass filter. Indeed, because $T_1(s)$ is a low-pass filter, $T_2(s) = \pm 4T_1(s)$ is also low-pass.

Furthemore note that

$$|T_2(j0)| = 4, \quad |T_2(j\bar{\omega})| = 2\sqrt{2}, \quad |T_2(j\omega_c)| = 2, \quad |T_2(j\infty)| = 0.$$ (+1 bonus point)

(e) [1 point (bonus)] Name one advantage and one disadvantage of your new circuit as compared with the circuit in Figure 4(b).

Solution:
You will get a point for saying anything meaninfull here. For example:
Advantage of $T_2(s)$: low-frequency response of $T_2(s)$ is similar to $T_1(s)$ but $T_2(s)$ attenuates high frequencies "faster" $|T_1(j\omega_c)| = 1/\sqrt{2}$ while $|T_2(j\omega_c)| = 1/2$! (+0.5 bonus point)
Disadvantage of $T_2(s)$: more components; (+0.5 bonus point)