MAE140 - Linear Circuits - Fall 10
Midterm, October 28

Instructions

(i) This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities

(ii) You have 70 minutes

(iii) Do not forget to write your name, student number, and instructor

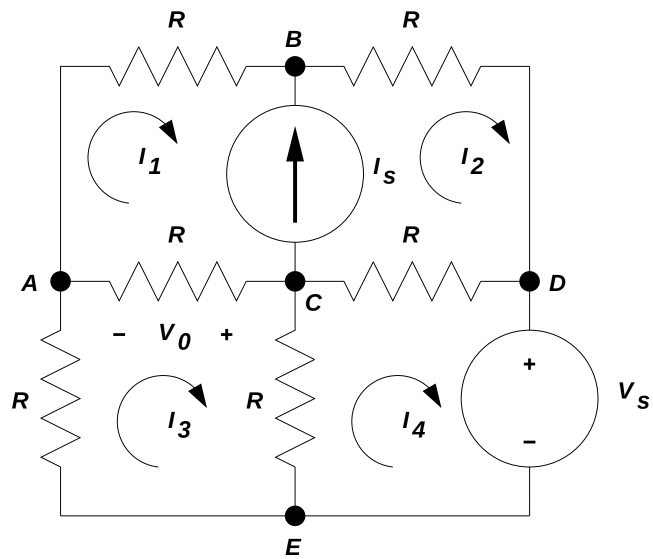


Figure 1: Circuit for all questions.

1. **Node Voltage Analysis**

[6 points] Formulate node-voltage equations for the circuit in Figure 1. Choose a ground node so that you do not have to write supernode equations. Use the node labels provided in the figure and clearly indicate the final equations, circuit unknowns and how you handle the presence of the voltage source. Also show how the voltage V_0 is related to the node voltages.

Do not modify the labels ~~or the location of the ground~~. No need to solve any equations!

Solution:

In order to deal with the voltage source without writing supernode equations, you can chose Node E or Node D as the ground node. We write the rest of the solution in terms of Node E here.

(+1 pt)

In this way, we can say that

$$V_D = V_S \tag{1}$$

and not write KCL at node D (this is method # 2). **(+1 pt)**

To find the values of the other voltages, we will use Kirchoff Current Law. Denote $G = 1/R$. Then, by inspection:

$$\begin{aligned} \text{Node A:} \quad & G(V_A - V_B) + G(V_A - V_C) + GV_A = 0 \\ \text{Node B:} \quad & G(V_B - V_A) + G(V_B - V_D) = I_S \\ \text{Node C:} \quad & G(V_C - V_A) + G(V_C - V_D) + GV_C = -I_S\,. \end{aligned}$$

The set of equations in V_A, V_B, V_C can be rewritten in matrix form as:

$$\begin{bmatrix} 3G & -G & -G \\ -G & 2G & 0 \\ -G & 0 & 3G \end{bmatrix} \begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix} = \begin{bmatrix} 0 \\ I_S + GV_S \\ -I_S + GV_S \end{bmatrix} \tag{2}$$

Equation (2) needs to be solved in the unknowns V_A, V_B, V_C whereas V_D is provided by Equation (1). **(+3 pts)**

Once we have solved for the node voltages, we can obtain V_O as

$$V_O = V_C - V_A.$$ **(+1 pt)**

2. **Mesh Current Analysis**
[6 points] Formulate mesh-current equations for the circuit in Figure 1. Use the mesh currents shown in the figure and clearly indicate the final equations, circuit unknowns and how you handle the presence of the current source. Also show how the voltage V_0 is related to the mesh currents.

Do not modify the circuit diagram or the labels. No need to solve any equations!

Solution:

The current source belongs to two meshes and, in this exam, the circuit cannot be rearranged to change this. Therefore, we will need combine Mesh 1 and Mesh 2 into one supermesh. **(+1 pt)**

This will give us two equations as follows:

$$\begin{aligned} \text{KVL applied to the supermesh:} \quad & RI_1 + R(I_1 - I_3) + RI_2 + R(I_2 - I_4) = 0, \\ \text{Supermesh constraint equation:} \quad & I_2 - I_1 = I_S\,. \end{aligned}$$

(+2 pts)

We also need to write KVL equations for meshes 3 and 4:

$$\begin{aligned} \text{Mesh 3:} \quad & RI_3 + R(I_3 - I_1) + R(I_3 - I_4) = 0, \\ \text{Mesh 4:} \quad & R(I_4 - I_3) + R(I_4 - I_2) + V_S = 0 \end{aligned}$$

The above four equations need to be solved for the four mesh currents I_1, I_2, I_3 and I_4. **(+2 pts)**

Once we have those, we can compute the voltage V_0 using Ohm's Law as

$$V_0 = R(I_1 - I_3).$$ **(+1 pt)**

3. **Linearity Analysis**
[6 points + 2 bonus points] For the circuit shown in Figure 1, use linearity to solve for the unkonw voltage V_0.

Solution:

We are more interested in your reasoning than your calculations. You will get all your points for explaining how you solve the problem, not for computing the solution.

We will use superposition to compute V_0 as the sum of two voltages

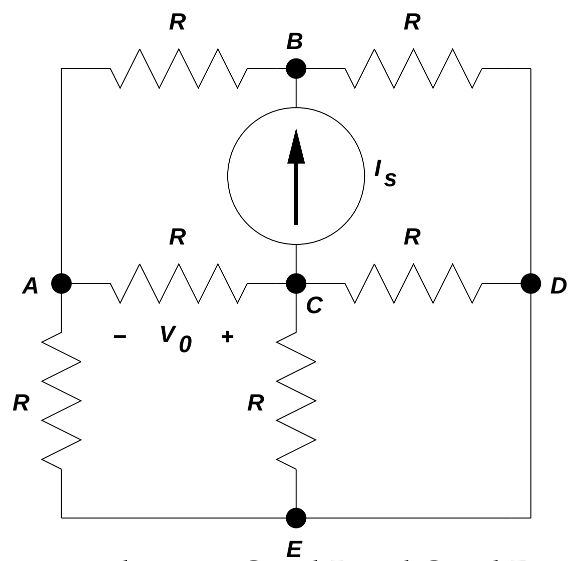
$$V_0 = V_0^1 + V_0^2$$

which we will compute for the following two circuits. **(+1 pt)**

In the first circuit, setting

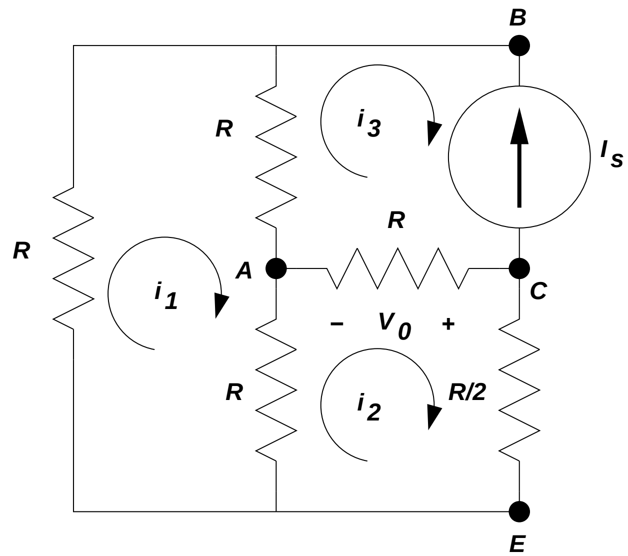
$$V_s = 0$$ **(+1 pt)**

leads to the diagram:



(+1 pt)

We can associate the two resistors between C and D and C and E in parallel and rearrange the circuit to avoid a supermesh as in the next diagram:



We will use method #2 not to write KVL at mesh 3 and obtain by inspection mesh-current equations:

$$\begin{bmatrix} 3R & -R & -R \\ -R & 2.5R & -R \end{bmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad\qquad i_3 = -I_s$$

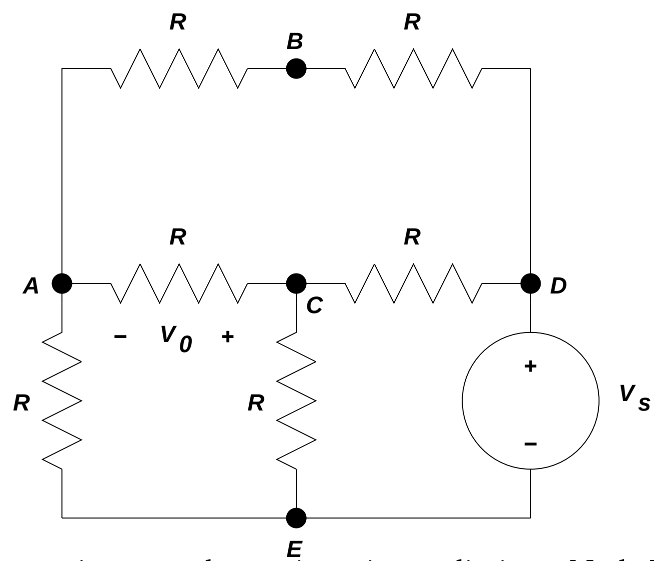
After you solve for i_1, i_2 and i_3 you should be able to compute

$$V_0^1 = R(i_3 - i_2)$$ **(+1/2 pt)**

In the second circuit, setting

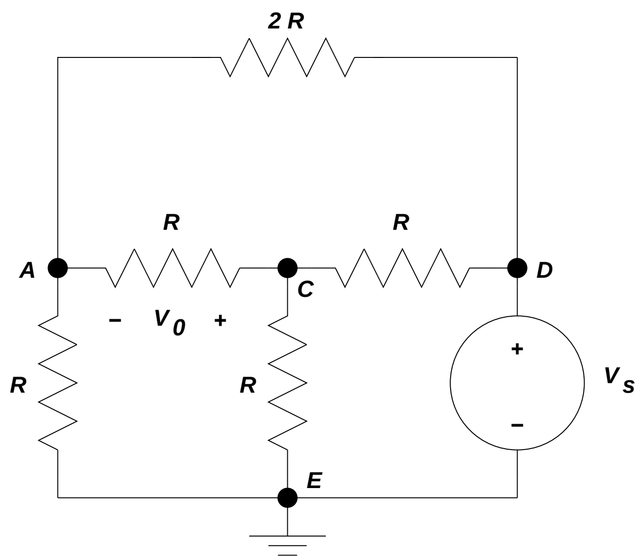
$$I_s = 0$$ **(+1 pt)**

leads to the diagram:



(+1 pt)

We can associate the two resistors on the top in series to eliminate Node B, choose the ground to be at node E leading to the diagram:



We use method #2 not to write KCL at node D and obtain the remaining node-voltage equations by inspection:

$$\begin{bmatrix} 2.5G & -G & -0.5G \\ -G & 3G & -G \end{bmatrix} \begin{pmatrix} V_A \\ V_C \\ V_D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad V_D = V_s$$

After you solve for V_A and V_C you should be able to compute

$$V_0^2 = V_C - V_A$$ **(+1/2 pt)**

If you care for the grueling computations...

In the first circuit, divide all by R and substitute i_3 to get

$$\begin{bmatrix} 3 & -1 \\ -1 & 2.5 \end{bmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} -I_s \\ -I_s \end{pmatrix} \qquad \Longrightarrow \qquad \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} -7/13 \\ -8/13 \end{pmatrix} I_s$$

then

$$V_0^1 = R(i_3 - i_2) = RI_s(8/13 - 1) = -(5/13)RI_s$$

In the second circuit, divide by G and substitute V_D to get

$$\begin{bmatrix} 2.5 & -1 \\ -1 & 3 \end{bmatrix} \begin{pmatrix} V_A \\ V_C \end{pmatrix} = \begin{pmatrix} 0.5V_s \\ V_s \end{pmatrix} \qquad \Longrightarrow \qquad \begin{pmatrix} V_A \\ V_C \end{pmatrix} = \begin{pmatrix} 5/13 \\ 6/13 \end{pmatrix} V_s$$

then

$$V_0^2 = V_C - V_A = (6/13 - 5/13)V_s = (1/13)V_s$$

Finally

$$V_0 = V_0^1 + V_0^2 = (1/13)V_s - (5/13)RI_s$$

If you got this far you deserve two more bonus points. One point if you got one correct partial answer, either V_0^1 or V_0^2. **(+2 bonus pts)**

4. **~~Nodal Analysis~~ Mesh Analysis with Dependent Source**
[2 bonus points] Suppose the voltage source in the circuit 1 is now a dependent voltage source with value $V_S = kV_0$. Show how the mesh-current equations you found in Problem 2 should be modified in order to take this dependency into account? Clearly indicate the final equations and circuit unknowns.

Use the results you obtained in Problem 1. No need to solve any equations!

Solution:

We should start by the solution of problem 2, which gives us the equations:

$$\begin{aligned}
2RI_1 + 2RI_2 - RI_3 - RI_4 &= 0, \\
-RI_1 \qquad + 3RI_3 - RI_4 &= 0, \\
-RI_2 - RI_3 + 2RI_4 + V_S &= 0 \\
I_2 - I_1 &= I_S,
\end{aligned}$$

The only equation that involves V_S is the third so we modify it to include the dependence on V_0:

$$-RI_2 - RI_3 + 2RI_4 + kV_0 = 0$$ **(+1 bonus pt)**

and use the expression for $V_0 = R(I_1 - I_3)$ also obtained in problem 2 to write

$$-RI_2 - RI_3 + 2RI_4 + kR(I_1 - I_3) = 0.$$

The final equations to be solved for I_1, I_2, I_3 and I_4 are

$$\begin{aligned}
2RI_1 + 2RI_2 \qquad - RI_3 \quad - RI_4 &= 0, \\
-RI_1 \qquad\qquad + 3RI_3 \quad - RI_4 &= 0, \\
kRI_1 \quad - RI_2 - (1+k)RI_3 + 2RI_4 &= 0, \\
I_2 - I_1 &= I_S
\end{aligned}$$ **(+1 bonus pt)**