MAE140 – Linear Circuits – Fall – 2012
Final, December 13th

Instructions

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.
2. You have 170 minutes.
3. Do not forget to write your name and student number.
4. This exam has 5 questions, for a total of 50 points and 6 bonus points.

1. **Circuit Equivalence**

   In this question you will find conditions for the circuits in Figures 1(a) and 1(b) to be equivalent. This is known as the $\Delta - Y$ equivalence. In the questions below, R_{AB} is the equivalent resistance as seen from terminals A and B, and so on. Answer the following questions:

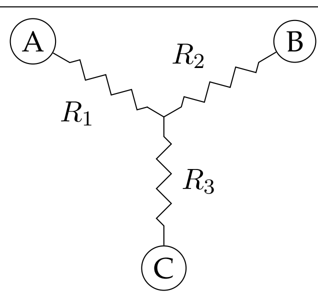
   (a) (3 points) For the circuit in Figure 1(a), show that

$$R_{AB} = R_1 + R_2, \qquad R_{BC} = R_2 + R_3, \qquad R_{AC} = R_1 + R_3.$$

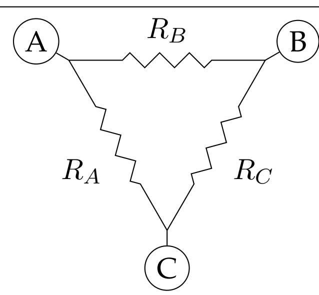
Solution: Calculating R_{AB}, the resistance R_3 sees no current and therefore can be taken out of the circuit. Consequently R_1 and R_2 are in series and $R_{AB} = R_1 + R_2$. (+1 point)
Calculating R_{BC}, the resistance R_1 sees no current and therefore can be taken out of the circuit. Consequently R_2 and R_3 are in series and $R_{BC} = R_2 + R_3$. (+1 point)
Calculating R_{AC}, the resistance R_2 sees no current and therefore can be taken out of the circuit. Consequently R_1 and R_3 are in series and $R_{AC} = R_1 + R_3$. (+1 point)

   (b) (3 points) For the circuit in Figure 1(b), show that

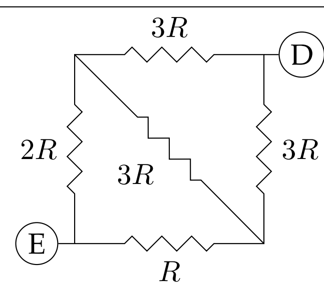
$$R_{AB} = \frac{R_B(R_A + R_C)}{R_A + R_B + R_C}, \qquad R_{BC} = \frac{R_C(R_A + R_B)}{R_A + R_B + R_C}, \qquad R_{AC} = \frac{R_A(R_B + R_C)}{R_A + R_B + R_C}.$$



(a) Y-circuit



(b) Δ-circuit



(c) Bridge circuit

Figure 1: Circuits for Question 1.

Solution: Calculating R_{AB}, the resistance R_B is in parallel with the series association of R_A and R_C, hence

$$R_{AB} = \frac{R_B(R_A + R_C)}{R_B + (R_A + R_C)} = \frac{R_B(R_A + R_C)}{R_A + R_B + R_C} \qquad \text{(+1 point)}$$

Calculating R_{AB}, the resistance R_C is in parallel with the series association of R_A and R_B, hence

$$R_{AB} = \frac{R_C(R_A + R_B)}{R_C + (R_A + R_B)} = \frac{R_C(R_A + R_B)}{R_A + R_B + R_C} \qquad \text{(+1 point)}$$

Calculating R_{AC}, the resistance R_A is in parallel with the series association of R_B and R_C, hence

$$R_{AB} = \frac{R_A(R_B + R_C)}{R_A + (R_B + R_C)} = \frac{R_A(R_B + R_C)}{R_A + R_B + R_C} \qquad \text{(+1 point)}$$

NOTE TO GRADER: Assign full points if a student works out in detail one case and correctly indicate the others.

(c) (2 points (bonus)) Show that if the Δ to Y formulas:

$$R_1 = \frac{R_A R_B}{R_A + R_B + R_C}, \qquad R_2 = \frac{R_B R_C}{R_A + R_B + R_C}, \qquad R_3 = \frac{R_A R_C}{R_A + R_B + R_C}$$

hold then the two circuits in Figures 1(a) and 1(b) are equivalent.

Hint: Note that in the Y circuit: $2R_1 = R_{AB} + R_{AC} - R_{BC}$.

Solution: From the Y circuit:

$$R_{AB} = R_1 + R_2, \qquad R_{BC} = R_2 + R_3, \qquad R_{AC} = R_1 + R_3.$$

and therefore

$$2R_1 = R_{AB} + R_{AC} - R_{BC}$$

which is the hint. From the Δ circuit:

$$\begin{aligned} 2R_1 &= R_{AB} + R_{AC} - R_{BC} \\ &= \frac{R_B(R_A + R_C)}{R_A + R_B + R_C} + \frac{R_A(R_B + R_C)}{R_A + R_B + R_C} - \frac{R_C(R_A + R_B)}{R_A + R_B + R_C} \\ &= \frac{2R_A R_B}{R_A + R_B + R_C}. \end{aligned}$$

Hence

$$R_1 = \frac{R_A R_B}{R_A + R_B + R_C}. \qquad \text{(+1 point)}$$

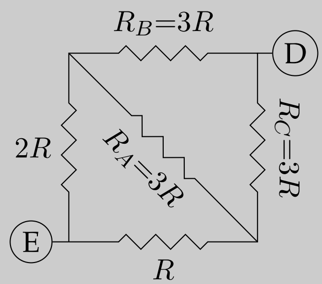
A similar proof works for the other two resistances because

$$2R_2 = R_{AB} + R_{BC} - R_{AC}, \qquad 2R_3 = R_{BC} + R_{AC} - R_{AB}. \qquad \text{(+1 point)}$$

NOTE TO GRADER: Assign full points if a student works out in detail one case and correctly indicate the others.

(d) (4 points) Use the Δ to Y formulas from part (c) to find the equivalent resistance as seen from terminals (D) and (E) in the bridge circuit of Figure 1(c).

Solution: Identify a Δ circuit in the original circuit, for example:

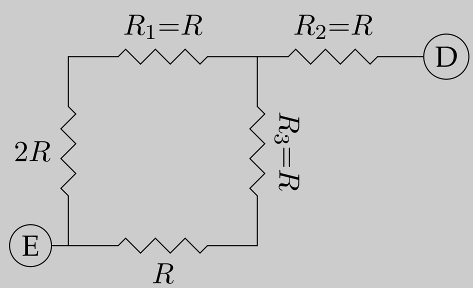


in which $R_A = R_B = R_C = 3R$. (+1 point)

The equivalent Y circuit is one where

$$R_1 = R_2 = R_3 = \frac{9R^2}{9R} = R.$$ (+1 point)

that is



(+1 point)

The equivalent resistance is therefore

$$\begin{aligned} R_{\text{eq}} &= R + 3R \parallel 2R \\ &= R + \frac{(3R) \times (2R)}{3R + 2R} \\ &= R + \frac{6R^2}{5R} \\ &= \frac{11}{5}R. \end{aligned}$$ (+1 point)

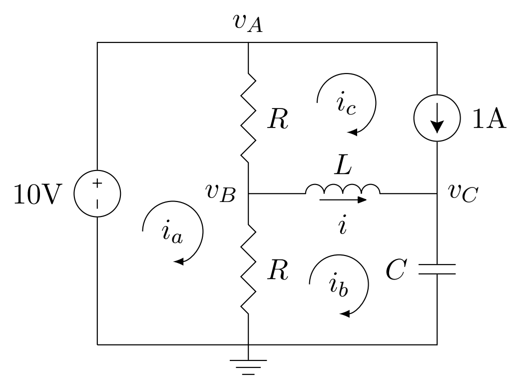


Figure 2: Circuit for Question 2.

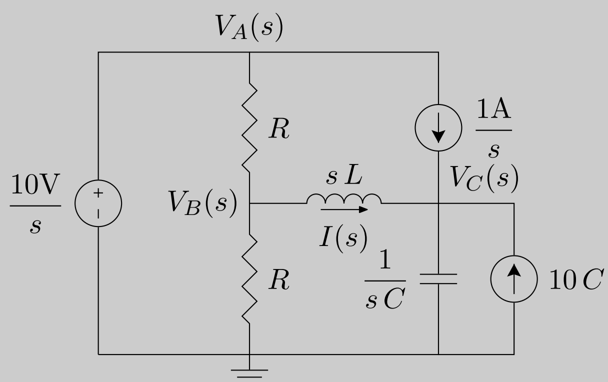
2. **Nodal and Mesh Analysis**

   In your answer to this problem, clearly indicate the voltages, currents and equations that need to be solved in each part. There is no need to actually solve any equations for this exam. All sources are constant. The initial current through the inductor is zero. The initial voltage across the capacitor is $v_C(0) = 10\text{V}$.

   (a) (4 points) Formulate node-voltage equations in the s-domain for the circuit in Figure 2. Use the reference node and other labels as shown in the figure. Use the initial conditions provided. Transform initial conditions into current sources.

Solution:

Transforming the circuit in the s-domain we obtain



(+2 points)

By inspection at nodes V_B and V_C:

$$\begin{bmatrix} -\frac{1}{R} & \frac{2}{R} + \frac{1}{s\,L} & -\frac{1}{s\,L} \\ 0 & -\frac{1}{s\,L} & \frac{1}{s\,L} + s\,C \end{bmatrix} \begin{pmatrix} V_A(s) \\ V_B(s) \\ V_C(s) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{0.1}{s} + 10\,C \end{pmatrix}$$

(+1 point)

along with the equation determining voltage $V_A(s)$:

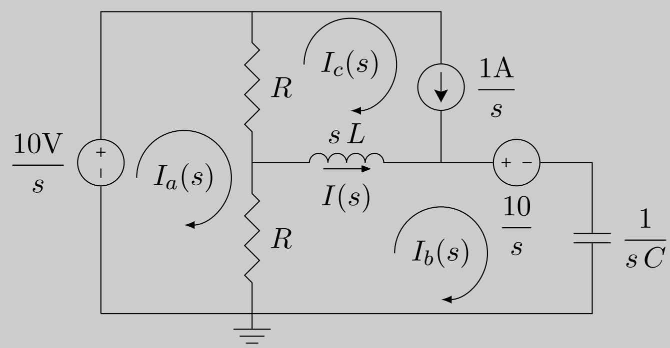
$$V_A(s) = \frac{10\text{V}}{s}.$$

(+1 point)

(b) (4 points) Formulate mesh-current equations in the s-domain for the circuit in Figure 2. Use the mesh currents shown in the figure. Use the initial conditions provided. Transform initial conditions into voltage sources.

Solution:

Transforming the circuit in the s-domain we obtain



(+2 points)

By inspection at meshes I_a and I_b:

$$\begin{bmatrix} 2R & -R & -R \\ -R & R + s\,L + \dfrac{1}{s\,C} & -s\,L \end{bmatrix} \begin{pmatrix} I_a(s) \\ I_b(s) \\ I_c(s) \end{pmatrix} = \begin{pmatrix} \dfrac{10}{s} \\ -\dfrac{10}{s} \end{pmatrix}$$ (+1 point)

along with the equation determining the mesh current $I_c(s)$:

$$I_c(s) = \frac{0.1\text{A}}{s}.$$ (+1 point)

(c) (2 points) Express the s-domain current $I(s)$, indicated in Figure 2 in the time domain as i, in terms of the s-domain node voltages and the mesh currents. You do not need to solve any equations.

Solution:

$$I(s) = \frac{V_B(s) - V_C(s)}{s\,L},$$ (+1 point)

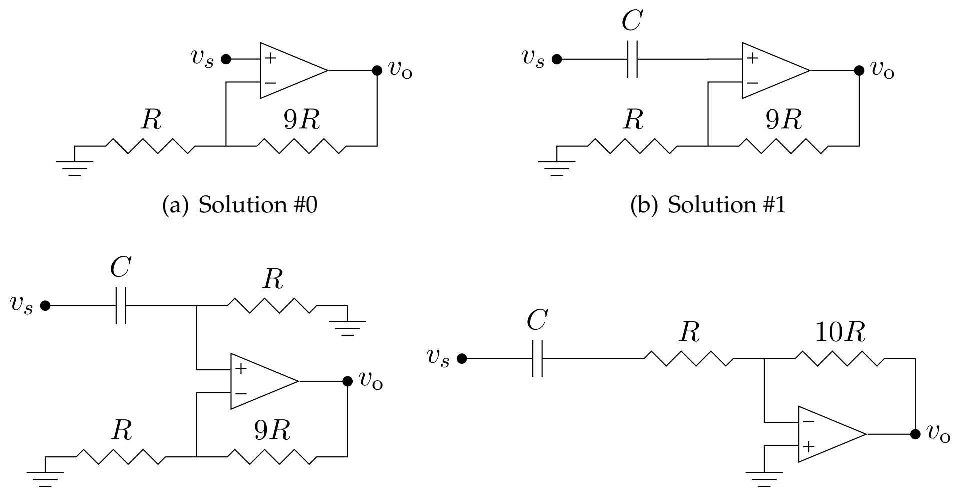
$$I(s) = I_b(s) - I_c(s).$$ (+1 point)

3. **Frequency Response and OpAmp Circuits**

A student built a transistor pre-amplifier to connect his guitar to a Digital Signal Processor (DSP). The input voltage to the DSP should be between -10V and 10V. The output of the student's pre-amplifier is

$$v_s(t) = 10\text{V} + v_x(t)$$

where $v_x(t)$ is the pre-amplified sound coming from his guitar. The signal $v_x(t)$ varies from -1V to 1V and only have components in frequencies higher than 20Hz. In order to maximize



(a) Solution #0 (b) Solution #1

(c) Solution #2 (d) Solution #3

Figure 3: Circuits for Question 3.

the resolution of the DSP, the student wants to build another amplifier using OpAmps that can amplify only the signal $v_x(t)$ by multiplying it by a factor of 10. That is, the output of the OpAmp circuit should be

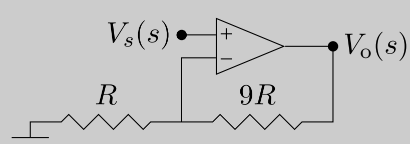
$$v_o(t) = 10\, v_x(t).$$

The student tried Solution #0 in Figure 3(b) but it did not work. A friend suggested that he added a capacitor to "block the DC component" from $v_s(t)$. That is Solution #1, which also did not work. Others friends suggested variations of this idea, which are Solution #2 and Solution #3. Answer the following questions for each of the circuits in Figure 3:

(a) (2 points) Assume zero initial conditions and convert the circuit into the s-domain.

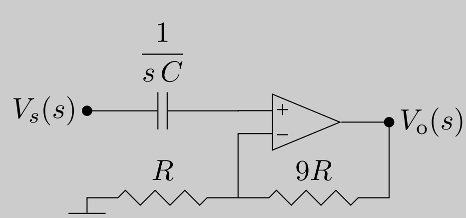
Solution: The circuits in the s-domain are:

Solution #0:



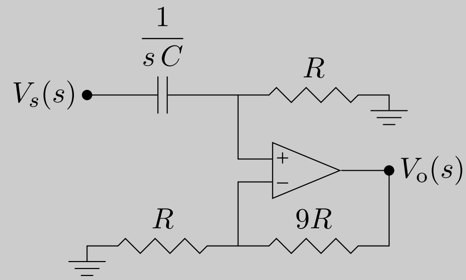
(+0.5 point)

Solution #1:



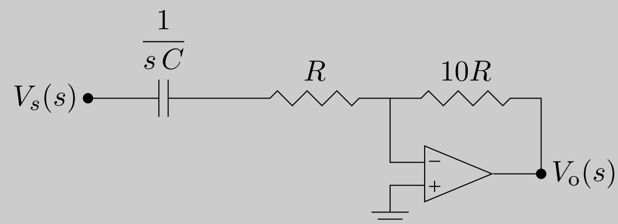
(+0.5 point)

Solution #2:



(+0.5 point)

Solution #3:



(+0.5 point)

(b) (2 points) Compute the output voltage $V_o(s)$ in terms of the input voltage $V_s(s)$ and the corresponding transfer function $G(s) = V_o(s)/V_s(s)$.

Solution: Solution #0. This is a standard non-inverting amplifier where

$$V_o(s) = \frac{9R+R}{R} V_s(s) = 10 V_s(s), \qquad G(s) = \frac{V_o(s)}{V_s(s)} = 10. \qquad \text{(+0.5 point)}$$

Solution #1. Because $I_+(s) = 0$ the voltage across the capacitor is zero and $V_+(s) = V_s(s)$. From this point on this is a standard non-inverting amplifier where

$$V_o(s) = \frac{9R+R}{R} V_+(s) = 10 V_s(s), \qquad G(s) = \frac{V_o(s)}{V_s(s)} = 10. \qquad \text{(+0.5 point)}$$

Solution #2. Because $I_+(s) = 0$ the voltage $V_+(s)$ can be computed using voltage division

$$V_+(s) = \frac{R}{R + \frac{1}{sC}} V_s(s) = \frac{s}{s + \frac{1}{RC}} V_s(s)$$

From this point on this is a standard non-inverting amplifier where

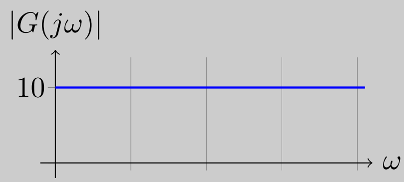
$$V_o(s) = \frac{9R+R}{R} V_+(s) = \frac{10\,s}{s + \frac{1}{RC}} V_s(s), \qquad G(s) = \frac{V_o(s)}{V_s(s)} = \frac{10\,s}{s + \frac{1}{RC}}. \qquad \text{(+0.5 point)}$$

Solution #3. This is a standard inverting amplifier where

$$V_o(s) = -\frac{10R}{R + \frac{1}{sC}} V_s(s) = \frac{-10\,s}{s + \frac{1}{RC}} V_s(s), \qquad G(s) = \frac{V_o(s)}{V_s(s)} = \frac{-10\,s}{s + \frac{1}{RC}}. \qquad \text{(+0.5 point)}$$

(c) (3 points) Sketch the magnitude plot of the frequency response, that is $|G(j\omega)|$, as a function of ω.

Solution: Solution #0 and #1 have the same transfer function $G(s) = 10$ which is constant for all frequencies, i.e. $|G(j\omega)| = 10$. A sketch of the response is:



(+1 point)

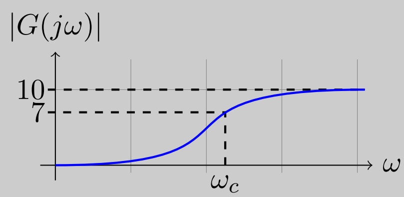
Solution #2 and #3 have transfer functions of the form

$$G(s) = \frac{\pm 10\, s}{s + \omega_c}, \qquad \omega_c = \frac{1}{RC},$$

which have the same magnitude response

$$|G(j\omega)| = \frac{10\,|\omega|}{\sqrt{\omega^2 + \omega_c^2}} \qquad \text{(+1 point)}$$

These are high-pass filters with cut-off frequency $\omega_c = 1/(RC)$ and gain equal to 10. A sketch of the response is:



(+1 point)

Then answer these questions for some bonus points:

(d) (1 point) Explain why Solution #0 does not work?

Solution: Because the DC part of v_s also gets amplified. The output would be something like:

$$v_o(t) = 100\text{V} + 10\, v_x(t),$$

which is not in the range ± 10V and will likely saturate the OpAmp. (+1 point)
NOTE TO GRADERS: Any reasonable answer earns a point.

(e) (1 point) Compute values for R and C such that the magnitude of the frequency response for Solutions #2 and #3 be greater than $10\,\sqrt{2}/2 \approx 7$ for any frequency component higher than 20Hz present in the signal v_x.

Solution: All that is needed to satisfy this requirement is that ω_c be low enough but not zero:

$$0 < \omega_c = \frac{1}{RC} \leq 2\pi 20 \qquad \text{(+.5 point)}$$

which implies

$$RC \geq \frac{1}{2\pi 20} \approx 0.008 = 8 \times 10^{-3}$$

For example

$$C = 1\text{nF} = 10^{-9}, \qquad R = 8\text{M}\Omega = 8 \times 10^{6}. \qquad \text{(+.5 point)}$$

NOTE TO GRADERS: Assign a full point if only mistake is forgetting to multiply by 2π.

(f) (1 point) Why Solutions #2 and #3 work?

Solution: Solutions #2 and #3 work because they have a zero gain at $\omega = 0$.
(+1 point)

In the s-domain, the response to the input signal v_s is equal to

$$V_\text{o}(s) = \frac{\pm 10\, j\omega}{j\omega + \omega_c} V_x(s).$$

If ω_c is low enough

$$V_\text{o}(s) \approx \pm 10 V_x(s) \qquad \Longrightarrow \qquad v_\text{o}(t) \approx \pm 10 v_x(t)$$

NOTE TO GRADERS: Any reasonable answer that mentions the highpass feature earns a point.

(g) (1 point (bonus)) Explain why Solution #2 works but Solution #1 does not work?

Solution: Solution #2 and #3 are very similar. However, because of the very high input impedance of the OpAmp, there is not enough current through the capacitor to "make it work" as a highpass filter in Solution #2. This is solved in Solution #3 by explicitely adding a path to ground through the resistor R. (+1 point)

NOTE TO GRADERS: Any reasonable answer earns a point.

4. **OpAmp Design**

In this question you will design an OpAmp circuit that solves a problem similar to the one in Question 3. Recall that the output of the pre-amplifier is of the form

$$v_s(t) = 10\text{V} + v_x(t)$$

where the signal $v_x(t)$ varies from -1V to 1V. In this question you will design an **OpAmp circuit using only resistors** that has as input $v_s(t)$ and produces as output

$$v_{\text{o}}(t) = 5\text{V} \pm 5\, v_x(t).$$

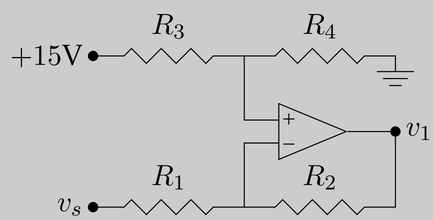
Assume that voltage supplies $+15$V and -15V are available. Do not add voltage or current sources to the circuits other than $v_s(t)$, $+15$V and -15V. Use voltage division or a gain if you need to produce a constant voltage from the ± 15V supplies. The symbol $\pm$ means it can be either positive or negative. Answer the following questions:

(a) (3 points) Design an OpAmp circuit that has $v_s(t)$ as input and produces

$$v_1(t) = \pm[1\text{V} \pm v_x(t)]$$

as its output.

Solution: One solution is to use the subtractor circuit:



(+1 point)

The output of this circuit is

$$\begin{aligned} v_1 &= \frac{R_1 + R_2}{R_1} \frac{R_4}{R_3 + R_4} 15\text{V} - \frac{R_2}{R_1} v_s \\ &= \left(\frac{R_1 + R_2}{R_1} \frac{R_4}{R_3 + R_4} 15\text{V} - \frac{R_2}{R_1} 10\text{V} \right) - \frac{R_2}{R_1} v_x. \end{aligned}$$

(+1 point)

In order to obtain

$$v_1 = -1\text{V} - v_x$$

we need to have

$$R_1 = R_2 \qquad \Longrightarrow \qquad \frac{R_2}{R_1} = 1 \quad \frac{R_1 + R_2}{R_1} = 2,$$

and

$$-1 = \frac{R_1 + R_2}{R_1} \frac{R_4}{R_3 + R_4} 15 - \frac{R_2}{R_1} 10 = \frac{R_4}{R_3 + R_4} 30 - 10$$

or

$$\frac{1}{1 + R_3/R_4} 30 = 9 \qquad \Longrightarrow \qquad \frac{R_3}{R_4} = \frac{10}{3} - 1 = \frac{7}{3}.$$

For example

$$R_1 = R_2 = R, \qquad R_3 = 7R, \qquad R_4 = 3R,$$ (+1 point)

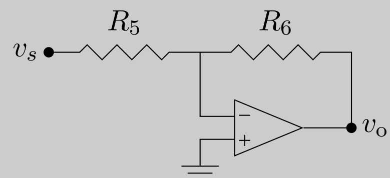
for some R, say 1MΩ.
NOTE TO GRADERS: Any correct answer earns the 3 points.

(b) (3 points) Design an OpAmp circuit that has $v_1(t)$ as input and produces

$$v_2(t) = \pm 5\, v_1(t)$$

as its output.

Solution:
Many easy solutions here. One is based on the inverting OpAmp



(+1 point)

The output of this circuit is

$$v_2 = -\frac{R_5}{R_6} v_1.$$ (+1 point)

In order to obtain

$$v_o = -5v_1$$

we need to have

$$R_6 = 5R_5 \qquad \Longrightarrow \qquad \frac{R_6}{R_5} = 5.$$

For example

$$R_5 = R, \qquad R_6 = 5R,$$ (+1 point)

for some R, say 1MΩ.
NOTE TO GRADERS: Any correct answer earns the 3 points.

(c) (4 points) Show how the circuits in parts (a)-(b) can be connected to produce

$$v_o(t) = 5\text{V} \pm 5\, v_x(t).$$

having as input $v_s(t)$.

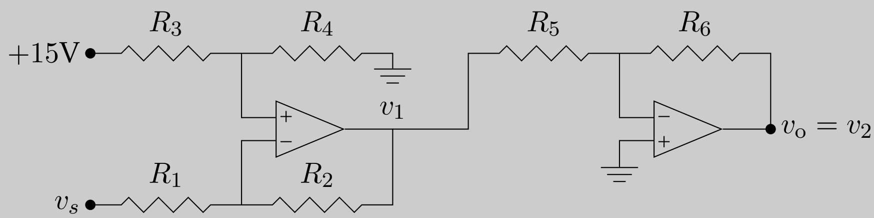
Solution: One possible solution is to simply connect

$$v_1 = -1\text{V} - v_x, \qquad v_\text{o} = v_2 = -5v_1 \qquad \text{(+1 point)}$$

which in this case leads to

$$v_\text{o} = -5(-1\text{V} - v_x) = 5\text{V} + 5\,v_x \qquad \text{(+1 point)}$$

The combined circuit looks like:



(+1 point)

where R_1, R_2, R_3, R_4, R_5, and R_6 are chosen as before, say:

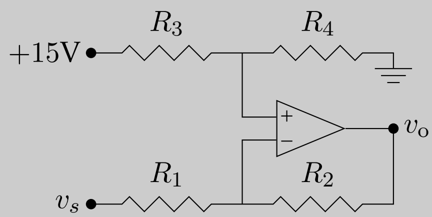
$$R_1 = R_2 = R, \qquad R_3 = 7R, \qquad R_4 = 3R, \qquad R_5 = R, \qquad R_6 = 5R, \qquad \text{(+1 point)}$$

for some R, say $1\text{M}\Omega$.

NOTE TO GRADERS: Any correct answer earns the 4 points.

(d) (2 points (bonus)) Show how the answer to this question can be implemented using a single OpAmp subtractor circuit (differential amplifier circuit) with one OpAmp.

Solution: Let us revisit the subtractor circuit solution:



The output of this circuit is

$$\begin{aligned} v_\text{o} &= \frac{R_1 + R_2}{R_1}\frac{R_4}{R_3 + R_4}15\text{V} - \frac{R_2}{R_1}v_s \\ &= \left(\frac{R_1 + R_2}{R_1}\frac{R_4}{R_3 + R_4}15\text{V} - \frac{R_2}{R_1}10\text{V}\right) - \frac{R_2}{R_1}v_x. \qquad \text{(+1 point)} \end{aligned}$$

In order to obtain

$$v_1 = 5\text{V} - 5v_x$$

we need to have

$$R_2 = 5R_1, \qquad \Longrightarrow \qquad \frac{R_2}{R_1} = 5, \quad \frac{R_1 + R_2}{R_1} = 6,$$

and

$$5 = \frac{R_1 + R_2}{R_1}\frac{R_4}{R_3 + R_4}15 - \frac{R_2}{R_1}10 = \frac{R_4}{R_3 + R_4}6 \times 15 - 50$$

or

$$\frac{1}{1 + R_3/R_4}6 \times 15 = 55 \qquad \Longrightarrow \qquad \frac{R_3}{R_4} = \frac{18}{11} - 1 = \frac{7}{11}.$$

For example

$R_1 = R_2 = R, \qquad R_3 = 7R, \qquad R_4 = 11R,$ (+1 point)

for some R, say $1\text{M}\Omega$.
NOTE TO GRADERS: Any correct answer earns the 2 points.

5. **Circuit with Switches**

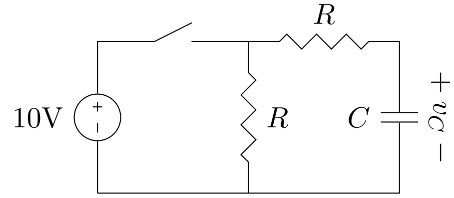
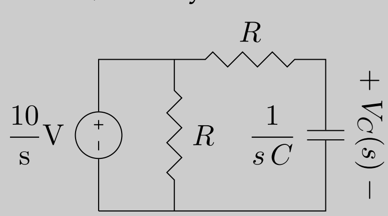


Figure 4: Circuit for Question 5.

In the circuit of Figure 4, the switch has been help open for a long time. At $t = 0$ the switch is closed and remains closed until $t = 10\text{s}$ when it is opened again. Assume $RC = 10\text{s}$ and answer the following questions:

(a) (1 point) Convert the circuit to the s-domain for $t \geq 0$. Clearly indicate how you handle the initial condition on the capacitor.

Solution:
After the switch is closed the circuit, already in the s-domain, becomes:



(+.5 point)

The initial voltage at the capacitor, $v_C(0)$, is zero because the switch has been open for a long time. (+.5 point)

(b) (4 points) Show that

$$v_C(t) = (1 - e^{-0.1t})10\text{V}, \qquad 0 \leq t \leq 10\text{s}.$$

Calculate $v_C(10\text{s})$. *Hint:* $1/e \approx 0.37$.

Solution:
The voltage at the capacitor, $V_C(s)$, can be computed using voltage division:

$$V_C(s) = \frac{\frac{1}{s\,C}}{R + \frac{1}{s\,C}} \frac{10}{s}\text{V} = \frac{\omega_c}{s+\omega_c}\frac{10}{s}\text{V}, \qquad \omega_c = \frac{1}{RC} = 0.1\text{s}^{-1}. \qquad \text{(+1 point)}$$

Expanding in partial fractions:

$$V_C(s) = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+0.1}$$

where

$$\alpha_1 = \lim_{s\to 0} \frac{1}{s+0.1} = 10\text{V}, \qquad \alpha_2 = \lim_{s\to -0.1} \frac{1}{s} = -10\text{V}.$$

we obtain

$$V_C(s) = \frac{10}{s} - \frac{10}{s+0.1} \qquad \text{(+1 point)}$$

Back to the t-domain:

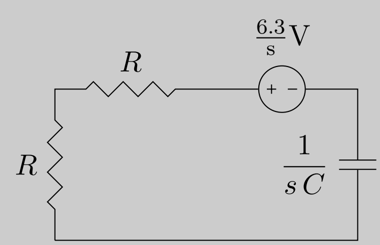
$$v_C(t) = (1 - e^{-0.1t})10\text{V}, \quad t \geq 0. \qquad \text{(+1 point)}$$

At $t = 10\text{s}$

$$v_C(10\text{s}) = (1 - e^{-1})10\text{V} \approx (1 - 3.7)10\text{V} = 6.3\text{V}. \qquad \text{(+1 point)}$$

(c) (1 point) Convert the circuit to the s-domain for $t \geq 10\text{s}$. Clearly indicate how you handle the initial condition on the capacitor.
Hint: Shift the origin of time to $t = 10$ and apply the Laplace transform as usual.

Solution:
We shift the origin to time to $t = 10\text{s}$ and with the switch opened derive the s-domain equivalent to the circuit:



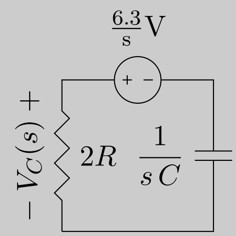
(+.5 point)
The initial voltage at the capacitor, $v_C(0) = 6.3\text{V}$, is obtained from $v_C(10)\text{s}$ computed in part (b). (+.5 point)

(d) (4 points) Show that

$$v_C(t) \approx e^{-0.05(t-10)} 6.3\text{V}, \qquad t \geq 10\text{s}.$$

Hint: If you used my hint in part (c) do not forget to shift your answer back in time.

Solution:
We associate the two resistors



(+1 point)

and compute $V_C(s)$ through voltage division:

$$V_C(s) = \frac{2R}{2R + \frac{1}{sC}} \frac{6.3}{s}\text{V} = \frac{10}{s + \omega_c}\text{V}, \qquad \omega_c = \frac{1}{2RC} = 0.05\text{s}^{-1}. \qquad \text{(+1 point)}$$

Back to the t-domain:

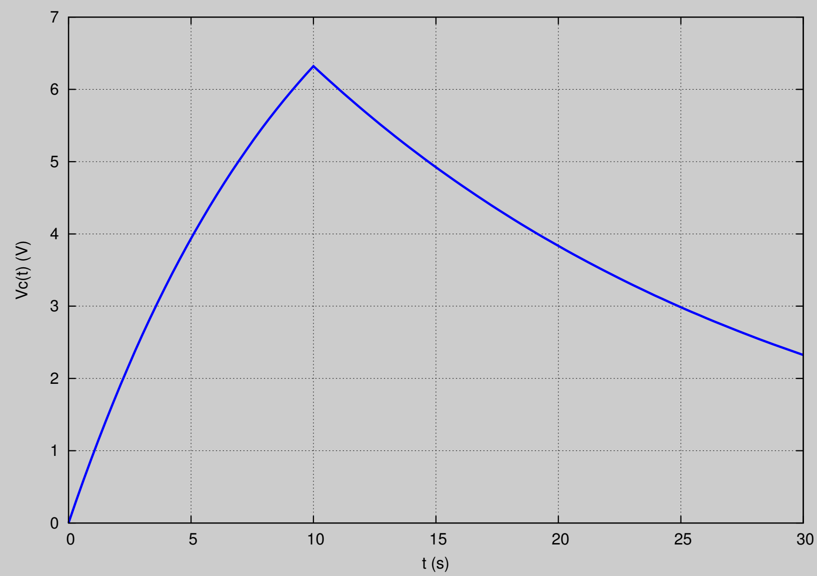
$$v_C(t) \approx e^{-0.05t} 6.3\text{V}, \quad t \geq 0. \qquad \text{(+1 point)}$$

Shifting back in time 10s:

$$v_C(t) \approx e^{-0.05(t-10)} 6.3\text{V}, \quad t \geq 10\text{s}. \qquad \text{(+1 point)}$$

(e) (1 point (bonus)) Sketch $v_C(t)$ for $t \geq 0$.

Solution:
A sketch of the solution $v_C(t)$ is shown below:



(+1 point)

Note the transition at $t = 10$s from one circuit to another.