**MAE140 - Linear Circuits - Fall 2012**
**Midterm, November 1st**

## Instructions

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities
2. You have 70 minutes
3. Write your answers on your "Blue Book"
4. Do not forget to write your name, student number, and instructor

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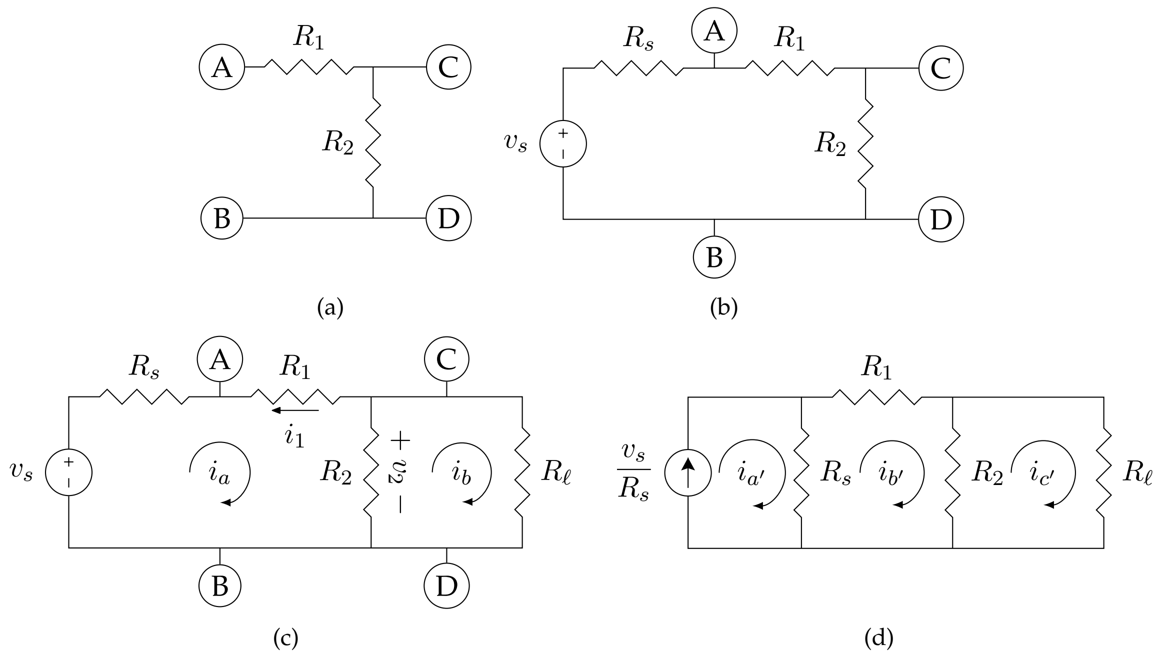


Figure 1: Circuits for Questions 1 and 2.

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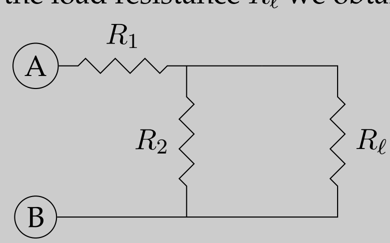
1. **Equivalent Circuits**

   The circuit in Fig. 1(a) is an *L-Pad Attenuator*. Answer the following questions:

   (a) (3 points) Show that when a (load) resistance $R_\ell$ is connected to terminals (C) and (D) the equivalent resistance as seen from terminals (A) and (B) is equal to

$$R_{AB} = \frac{R_1(R_2 + R_\ell) + R_2 R_\ell}{R_2 + R_\ell}.$$

**Solution:** After connecting the load resistance $R_\ell$ we obtain the circuit



in which $R_2$ is in parallel with $R_\ell$. (+1 point)

The parallel association of $R_2$ and $R_\ell$ is then in series with $R_1$. (+1 point)

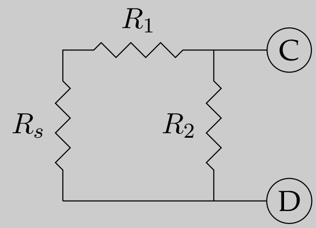
Hence

$$R_{AB} = R_1 + \frac{1}{\frac{1}{R_2} + \frac{1}{R_\ell}} = R_1 + \frac{R_2 R_\ell}{R_2 + R_\ell} = \frac{R_1(R_2 + R_\ell) + R_2 R_\ell}{R_2 + R_\ell}. \qquad \text{(+1 point)}$$

(b) (3 points) Show that when a (source) resistance $R_s$ is connected to terminals (A) and (B) the equivalent resistance as seen from terminals (C) and (D) is equal to

$$R_{CD} = \frac{R_2(R_1 + R_s)}{R_2 + R_1 + R_s}.$$

**Solution:** After connecting the source resistance $R_s$ we obtain the circuit




in which $R_1$ is in series with $R_s$. (+1 point)

The series association of $R_1$ and $R_s$ is then in parallel with $R_2$. (+1 point)

Hence

$$R_{CD} = \frac{1}{\frac{1}{R_2} + \frac{1}{R_1 + R_s}} = \frac{R_2(R_1 + R_s)}{R_2 + R_1 + R_s}. \qquad \text{(+1 point)}$$

(c) (2 points (bonus)) Verify that when a (source) resistance $R_s$ is connected to terminals (A) and (B) and

$$R_2 = \frac{R_\ell(R_1 + R_s)}{R_1 + R_s - R_\ell}, \qquad R_1 \geq R_\ell - R_s, \qquad R_1 \geq 0$$

then $R_{CD} = R_\ell$.

**Solution:** Substitute the given formula for $R_2$ into $R_{CD}$ calculated in part (b):

$$\begin{aligned} R_{CD} &= \frac{\frac{R_\ell(R_1+R_s)^2}{R_1+R_s-R_\ell}}{\frac{R_\ell(R_1+R_s)}{R_1+R_s-R_\ell} + R_1 + R_s} \qquad \text{(+1 point)} \\ &= \frac{R_\ell(R_1 + R_s)^2}{R_\ell(R_1 + R_s) + (R_1 + R_s)(R_1 + R_s - R_\ell)} \\ &= \frac{R_\ell(R_1 + R_s)}{R_\ell + R_1 + R_s - R_\ell} = R_\ell \end{aligned}$$

Plus one point for correct conclusion. (+1 point)

2. **Thévenin Equivalence and Voltage Division**

   Connect a voltage source $v_s$ with resistance $R_s$ to terminals (A) and (B) as shown in Fig. 1(b). Answer the following questions:

   (a) (2 points) Use voltage division to show that the Thévenin equivalent circuit as seen from terminals (C) and (D) has resistance and voltage

   $$R_T = \frac{R_2(R_1 + R_s)}{R_1 + R_2 + R_s}, \qquad v_T = K\, v_s, \qquad K = \frac{R_T}{R_1 + R_s}.$$

   $K$ is the circuit's *voltage attenuation*. *Hint: Use the answers to Question 1.*

   **Solution:** When the voltage source is $v_s = 0$ we obtain the circuit of Question 1 part (b). Hence

   $$R_T = R_{CD} = \frac{R_2(R_1 + R_s)}{R_1 + R_2 + R_s}. \qquad \text{(+0.5 point)}$$

   The Thévenin voltage is obtained from the open-circuit as in Fig. 1(b) through voltage-division as

   $$v_T = \frac{R_2}{R_1 + R_2 + R_s} v_s. \qquad \text{(+1 point)}$$

   Note that

   $$v_T = \frac{R_2(R_1 + R_s)}{R_1 + R_2 + R_s} \frac{1}{R_1 + R_s} v_s = \frac{R_T}{R_1 + R_s} v_s = K\, v_s. \qquad \text{(+0.5 point)}$$

   NOTE TO GRADERS: Students may also derive the equivalence through open-circuit and short-circuit calculations.

   (b) (2 points) When $R_1$ and $R_2$ are selected as in part (c) of Question 1, that is

   $$R_2 = \frac{R_\ell(R_1 + R_s)}{R_1 + R_s - R_\ell}, \qquad R_1 \geq R_\ell - R_s, \qquad R_1 \geq 0,$$

   then

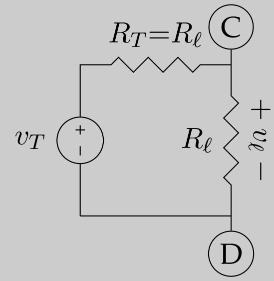
   $$R_T = R_\ell, \qquad K = \frac{R_\ell}{R_1 + R_s}.$$

   With such values of $R_1$ and $R_2$, connect a load resistance $R_\ell$ to terminals (C) and (D) as shown in Fig. 1(c). Show that the power absorbed by the load is

   $$p_\ell = \frac{K^2}{4} p_0, \qquad p_0 = \frac{v_s^2}{R_\ell}.$$

   $K^2$ is the circuit's *power attenuation*. *Hint: Use part (a) and voltage division.*

   **Solution:** After connecting the load $R_\ell$ and substituting the circuit on the left of the terminals (C) and (D) by its Thévenin equivalent we obtain the voltage divider:




(+1 point)

The voltage across the load resistance $R_\ell$ can be computed recalling that $R_T = R_\ell$ and using voltage division:

$$v_\ell = \frac{R_\ell}{R_\ell + R_\ell} v_T = \frac{1}{2} v_T = \frac{K}{2} v_s. \qquad \text{(+0.5 point)}$$

The power absorbed by the load is then

$$p_\ell = \frac{v_\ell^2}{R_\ell} = \frac{K^2}{4} \frac{v_s^2}{R_\ell} = \frac{K^2}{4} p_0. \qquad \text{(+0.5 point)}$$

(c) (2 points) Calculate values for the resistors $R_1 \geq 0$ and $R_2 \geq 0$ to design an L-Pad Attenuator in which $R_T = R_\ell = 50\Omega$ (as in a coaxial input connector) is connected to a voltage source with resistance $R_s = 300\Omega$ (as in a VHF or FM radio antenna) and achieve a voltage attenuation $K = 0.1$. *Hint: Use the expressions in part (b).*

**Solution:** From $K = 1/10$, $R_\ell = 50\Omega$, $R_s = 300\Omega$ we determine

$$\frac{1}{10} = \frac{50}{R_1 + 300} \qquad \Longrightarrow \qquad R_1 = 500 - 300 = 200\Omega \qquad \text{(+1 point)}$$

and then

$$R_2 = \frac{50(200 + 300)}{200 + 300 - 50} = \frac{2500}{45} \approx 56\Omega. \qquad \text{(+1 point)}$$

(d) (2 points (bonus)) Use the resistances $R_1$ and $R_2$ you calculated in part (c) to compute the L-Pad input resistance $R_{AB}$ and output resistance $R_{CD}$ as in Question 1 when $R_\ell = 50\Omega$ and $R_s = 300\Omega$. In which way does the circuit match the source and load resistances?

**Solution:** Substituting $R_1$ and $R_2$ we obtain

$$R_{AB} = 200 + \frac{56 \times 50}{56 + 50} \approx 226\Omega, \qquad R_{CD} = \frac{56(200 + 300)}{56 + 200 + 300} \approx 50\Omega. \qquad \text{(+1 point)}$$

The circuit makes the Thévenin resistance $R_T = R_{CD} \approx 50\Omega$ as seen from C and D be very close to the load $50\Omega$ resistance. It makes the resistance $R_{AB} \approx 226\Omega$ as seen from A and B be close but not equal to the source $300\Omega$ resistance. (+1 point)

3. **Mesh Current Analysis**

Consider the circuits in Fig. 1(c) and Fig. 1(d).

(a) (3 points) Formulate mesh-curent equations for the circuit in Fig. 1(c). Use the mesh currents and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns.

**Solution:** Writing the KVL equations by inspection in terms of the mesh currents we obtain the current equations:

$$\begin{bmatrix} R_s + R_1 + R_2 & -R_2 \\ -R_2 & R_\ell + R_2 \end{bmatrix} \begin{pmatrix} i_a \\ i_b \end{pmatrix} = \begin{pmatrix} v_s \\ 0 \end{pmatrix} \qquad \text{(+2 points)}$$

which need to be solved in terms of the mesh currents $i_a$ and $i_b$. (+1 point)

(b) (2 points) Show how the current $i_1$ and the voltage $v_2$ are related to the mesh currents.

**Solution:** The current $i_1$ is obtained directly from the mesh current $i_a$ as

$$i_1 = -i_a. \qquad \text{(+1 point)}$$

The voltage $v_2$ is obtained after using Ohm's law

$$v_2 = R_2(i_a - i_b). \qquad \text{(+1 point)}$$

(c) (1 point) Without writing any equations, show how the mesh currents in the circuit in Fig. 1(d) are related to the mesh currents in the circuit in Fig. 1(c). Calculate $i_{a'}$, $i_{b'}$ and $i_{c'}$ as a function of $i_a$ and $i_b$.

**Solution:** The circuits are equivalent because the current source is obtained from the voltage source through a source transformation. As a results of this equivalence

$$i_{b'} = i_a, \qquad i_{c'} = i_b, \qquad i_{a'} = \frac{v_s}{R_s}. \qquad \text{(+1 point)}$$