MAE140 – Linear Circuits – Fall – 2015
Final, December 8th

Instructions

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.
2. You have 170 minutes.
3. Do not forget to write your name and student number.
4. This exam has 5 questions, for a total of 60 points and 1 bonus points.

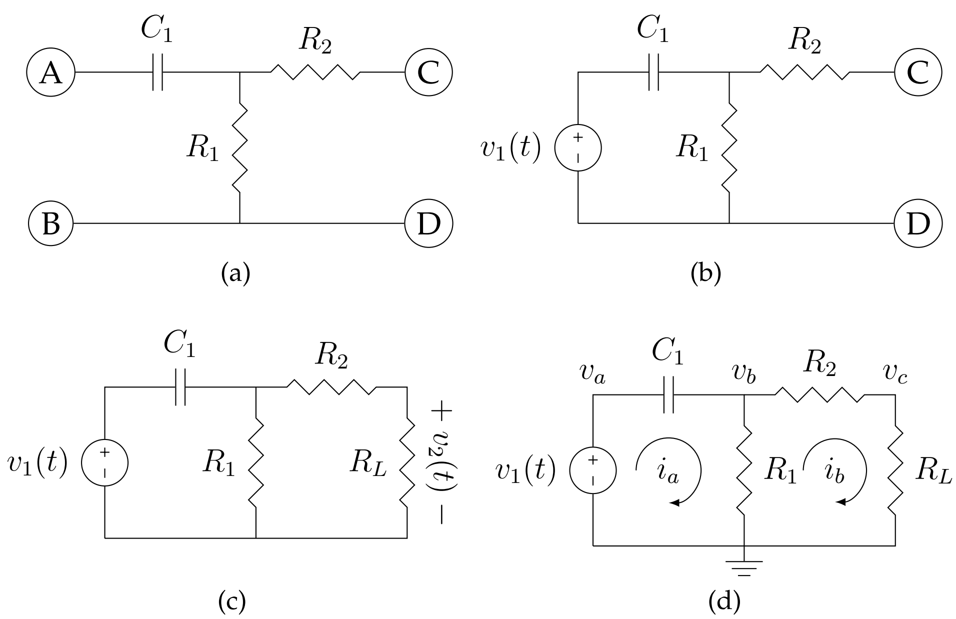


Figure 1: Circuits for Questions 1–4

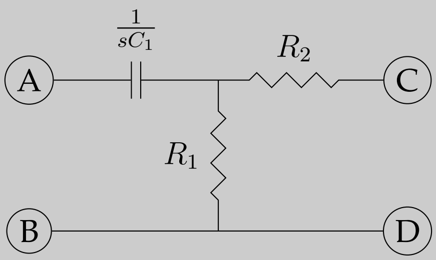
1. **Circuit Equivalence**

IMPORTANT: All impedances should be given as a ratio of two polynomials!

(a) (3 points) Assume zero-initial conditions and find the s-domain equivalent circuit as seen from terminals (A) and (B) in Fig. 1a.

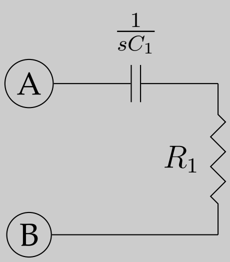
Solution:

First convert the circuit to the s-domain to obtain:



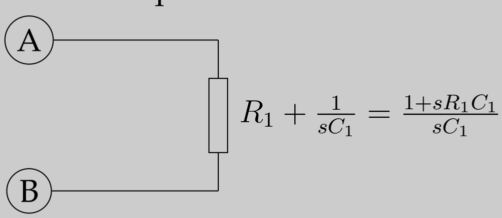
(+1 point)

As seen from (A) to (B) the circuit is simply:



(+1 point)

which is equivalent to:



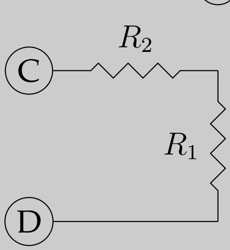
(+1 point)

(b) (2 points) Assume zero-initial conditions and find the s-domain equivalent circuit as seen from terminals (C) and (D) in Fig. 1a.

Solution:

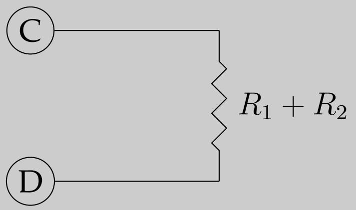
The circuit has already been converted to the s-domain in part (a).

As seen from (C) to (D) the circuit is simply:



(+1 point)

which is equivalent to:



(+1 point)

(c) (5 points) Assume zero initial conditions, connect a voltage source $v_1(t)$ to the terminals Ⓐ and Ⓑ as in Fig. 1b and show that the resulting Thevenin equivalent circuit as seen from terminals Ⓒ and Ⓓ has voltage

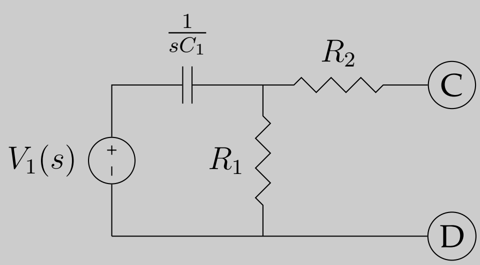
$$V_T(s) = \frac{sR_1C_1}{1 + sR_1C_1} V_1(s) \tag{1}$$

and impedance

$$Z(s) = \frac{R_1 + R_2 + sR_2R_1C_1}{1 + sR_1C_1}. \tag{2}$$

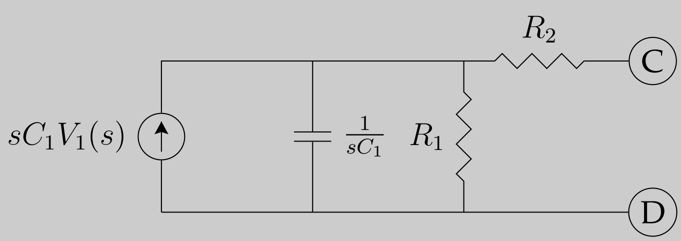
Solution:

Start again by converting to the s-domain:



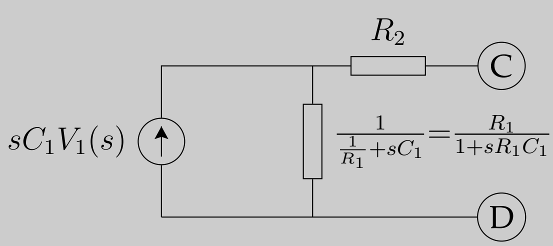
(+1 point)

After a source transformation:



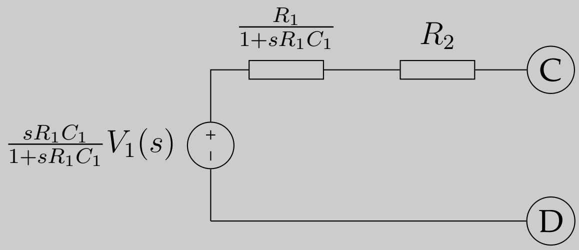
(+1 point)

Associating the capacitor and resistor in parallel:



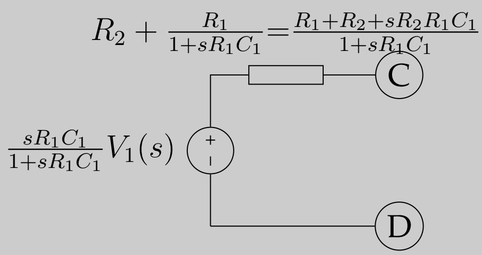
(+1 point)

Another source transformation produces:



(+1 point)

and finally:



(+1 point)

2. **Short-Circuit Protection**

   Consider the circuit in Fig. 1b. Short-circuit the terminals C and D and answer the following questions:

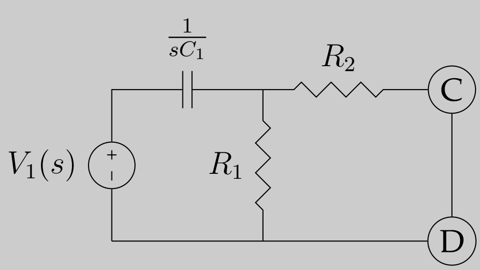
   (a) (2 points) Show that the transfer-function $T(s) = I_2(s)/V_1(s)$, where i_2 is the current through the resistor R_2 is:

$$T(s) = \frac{sR_1C_1}{R_1 + R_2 + sR_2R_1C_1}$$

Hint: Use the answers to Problem 1.

Solution:

Short-circuit C and D:



(+1/2 point)

to notice that the current in R_2 is the short-circuit current, which can be obtained directly from the Thevenin equivalent from Problem 1 as:

$$I_2(s) = \frac{V_T(s)}{Z(s)} = \frac{\frac{sR_1C_1}{1+sR_1C_1}}{\frac{R_1+R_2+sR_2R_1C_1}{1+sR_1C_1}}V_1(s) = \frac{sR_1C_1}{R_1 + R_2 + sR_2R_1C_1}V_1(s)$$ (+1 point)

and

$$T(s) = \frac{I_2(s)}{V_1(s)} = \frac{sR_1C_1}{R_1 + R_2 + sR_2R_1C_1} \qquad \text{(+1/2 point)}$$

(b) (3 points) Calculate and sketch $|T(j\omega)|$.

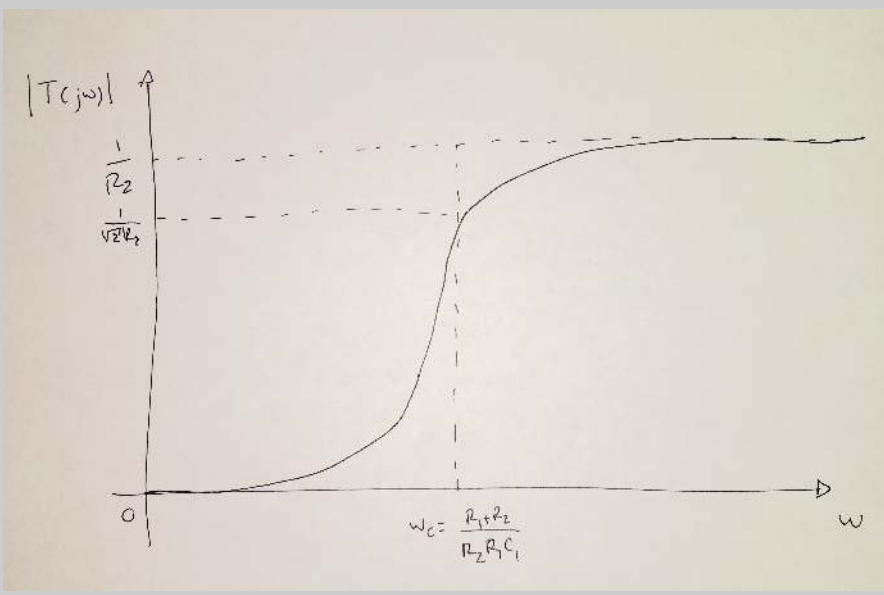
Solution:

$$\begin{aligned} |T(j\omega)| &= \frac{|j\omega R_1C_1|}{|R_1 + R_2 + j\omega R_2R_1C_1|} \\ &= \frac{1}{R_2}\frac{|j\omega|}{|\frac{R_1+R_2}{R_2R_1C_1} + j\omega|} \end{aligned} \qquad \text{(+1 point)}$$

This is a high-pass filter since at $\omega = 0$ the value $|T(j0)| = 0$ and at $\omega \to \infty$

$$\lim_{\omega\to\infty} |T(j\omega)| = \frac{1}{R_2} \qquad \text{(+1 point)}$$

The cutoff frequency is $\omega_c = (R_1 + R_2)/(R_2R_1C_1)$ and a sketch of the $|T(j\omega)|$ is:



(+1 point)

(c) (4 points) Calculate R_2 so that the maximum steady-state amplitude of the current $i_2(t)$ when the input voltage is $v_1(t) = 5\cos(\omega t)$V be less than 500mA for any possible frequency ω.

Solution:
When $v_1(t) = 5\cos(\omega t)$ the steady-state value of the current $i_2(t)$ is calculated

using the frequency response:

$$i_2^{SS}(t) = 5|T(j\omega)|\cos(\omega t + \angle T(j\omega)) \qquad \text{(+1 point)}$$

The amplitude of this current is

$$|i_2^{SS}(t)| = 5|T(j\omega)| \qquad \text{(+1 point)}$$

Because $T(j\omega)$ behaves as a high-pass filter its maximum is at $\omega \to \infty$ so that

$$|i_2^{SS}(t)| \leq 5\max_{\omega}|T(j\omega)| = \frac{5}{R_2}. \qquad \text{(+1 point)}$$

In order for this amplitude to be less than 500mA

$$\frac{5}{R_2} \leq 0.5 \qquad \Longrightarrow \qquad R_2 \geq 10\Omega \qquad \text{(+1 point)}$$

(d) (1 point) Explain in which way R_2 "protects" the circuit in Fig. 1b against a short-circuit.

Solution:
Because of the answer to part (c) the current provided by the circuit is limited by R_2. If R_2 were to be zero there would be no limit in the current. (+1 point)

3. **AC coupling**

Consider the circuit in Fig. 1c. Assume that $R_L \gg R_2$.

(a) (2 points) Show that the transfer function $T(s) = V_2(s)/V_1(s)$ is approximately:

$$T(s) = \frac{RC_1 s}{1 + RC_1 s}, \qquad R = \frac{R_1 R_L}{R_1 + R_L}$$

Solution:
If $R_L \gg R_2$ then R_1 and R_L are in parallel with value

$$R = \frac{R_1 R_L}{R_1 + R_L} \qquad \text{(+1 point)}$$

and the transfer-function $T(s) = V_2(s)/V_1(s)$ can be calculated by voltage division from:

$$V_2(s) = \frac{R}{R + \frac{1}{sC_1}} V_1(s) = \frac{sRC_1}{1 + sRC_1} V_1(s) = T(s)V_1(s). \qquad \text{(+1 point)}$$

IMPORTANT: For the rest of this question set $R_L = R_1$ and $R_1 C_1 = 0.02\text{s}$.

(b) (3 points) Use the frequency response method or a direct calculation to compute the steady-state response of the circuit to an input of the form $v_1(t) = 5$V.

Solution:
If $R_L = R_1$ then

$$R = \frac{R_1}{2}, \qquad T(s) = \frac{R_1 C_1 s}{2 + R_1 C_1 s} = \frac{0.02s}{2 + 0.02s} = \frac{s}{100 + s} \qquad \text{(+2 point)}$$

Using the frequency response method to calculate the steady state response to a constant input $v_1(t) = 5$V we obtain

$$v_2^{SS}(t) = 5\,T(j0) = 0. \qquad \text{(+1 point)}$$

(c) (3 points) Use the frequency response method or a direct calculation to compute the steady-state response of the circuit to an input of the form $v_1(t) = 5\cos(\omega t)$V where $\omega = 10{,}000$rad/s.

Solution:
Using the frequency response method to calculate the steady state response to an input $v_1(t) = 5\cos(\omega t)$V we obtain

$$v_2^{SS}(t) = 5|T(j\omega)|\cos(\omega t + \angle T(j\omega)). \qquad \text{(+1 point)}$$

Substituting $\omega = 10{,}000$ we calculate

$$T(j10\,000) = \frac{j10{,}000}{100 + j10{,}000} = \frac{j100}{1 + j100} \approx 0.9999 + j0.0099 \approx 0.99 \qquad \text{(+1 point)}$$

from which

$$v_2^{SS}(t) \approx 0.9999 \times 5\cos(\omega t) = 4.99\cos(\omega t)\text{V}. \qquad \text{(+1 point)}$$

(d) (2 points) Use linearity to calculate the steady-state response of the circuit to an input of the form $v_1(t) = 5 + 5\cos(\omega t)$ where $\omega = 10{,}000$rad/s.

Solution:
Because

$$v_1(t) = v_1^1(t) + v_1^2(t), \qquad v_1^1(t) = 5, \qquad v_1^2(t) = 5\cos(\omega t) \qquad \text{(+1 point)}$$

we can use the answers to parts (b) and (c) to calculate

$$v_2^{SS}(t) = 0 + 4.99\cos(\omega t)\text{V}. \qquad \text{(+1 point)}$$

(e) (1 point (bonus)) Explain why the circuit in Fig. 1a is known as an *AC Coupler*.

Solution:

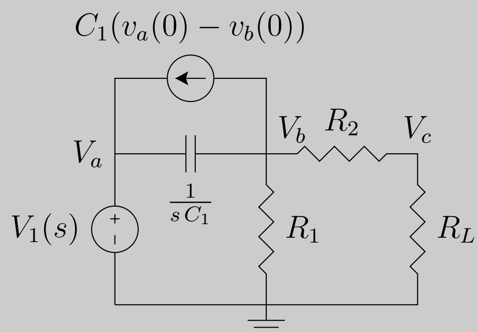
Because it "blocks" the DC component, the 5V input in this problem, and let the AC component, $\omega \gg 0$, pass through the circuit. (+1 point)

4. **Circuit Analysis**

(a) (5 points) Convert the circuit in Fig. 1d to the s-domain and formulate its node-voltage equations. Use the node-voltage and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns. Do not assume zero initial conditions. Make sure your final equations only involve node-voltages.

Solution:

Convert to the s-domain taking into account the initial conditions



(+2 points)

Method #2 provides the following equations obtained by inspection:

$$V_a(s) = V_1(s)$$

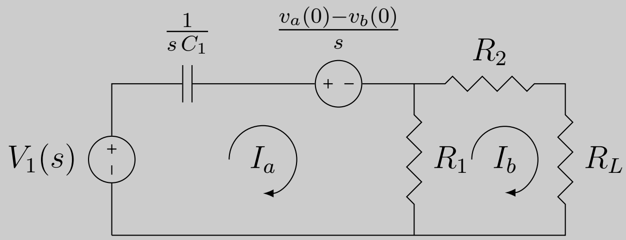
$$\begin{bmatrix} -sC_1 & sC_1 + \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} \\ 0 & -\frac{1}{R_2} & \frac{1}{R_2} + \frac{1}{R_L} \end{bmatrix} \begin{pmatrix} V_a(s) \\ V_b(s) \\ V_c(s) \end{pmatrix} = \begin{pmatrix} C(v_a(0) - v_b(0)) \\ 0 \end{pmatrix} \quad \text{(+2 points)}$$

which should be solved for the unknowns $V_a(s)$, $V_b(s)$ and $V_c(s)$. (+1 point)

(b) (5 points) Convert the circuit in Fig. 1d to the s-domain and formulate its mesh-current equations. Use the mesh-currents and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns. Do not assume zero initial conditions. Make sure your final equations only involve mesh-currents.

Solution:

Convert to the s-domain taking into account the initial conditions



(+2 points)

Writing the equations by inspection:

$$\begin{bmatrix} R_1 + \frac{1}{sC_1} & -R_1 \\ -R_1 & R_1 + R_2 + R_L \end{bmatrix} \begin{pmatrix} I_a(s) \\ I_b(s) \end{pmatrix} = \begin{pmatrix} V_1(s) - \frac{v_a(0)-v_b(0)}{s} \\ 0 \end{pmatrix}$$

(+2 points)

which should be solved for the unknowns $I_a(s)$, $I_b(s)$. (+1 point)

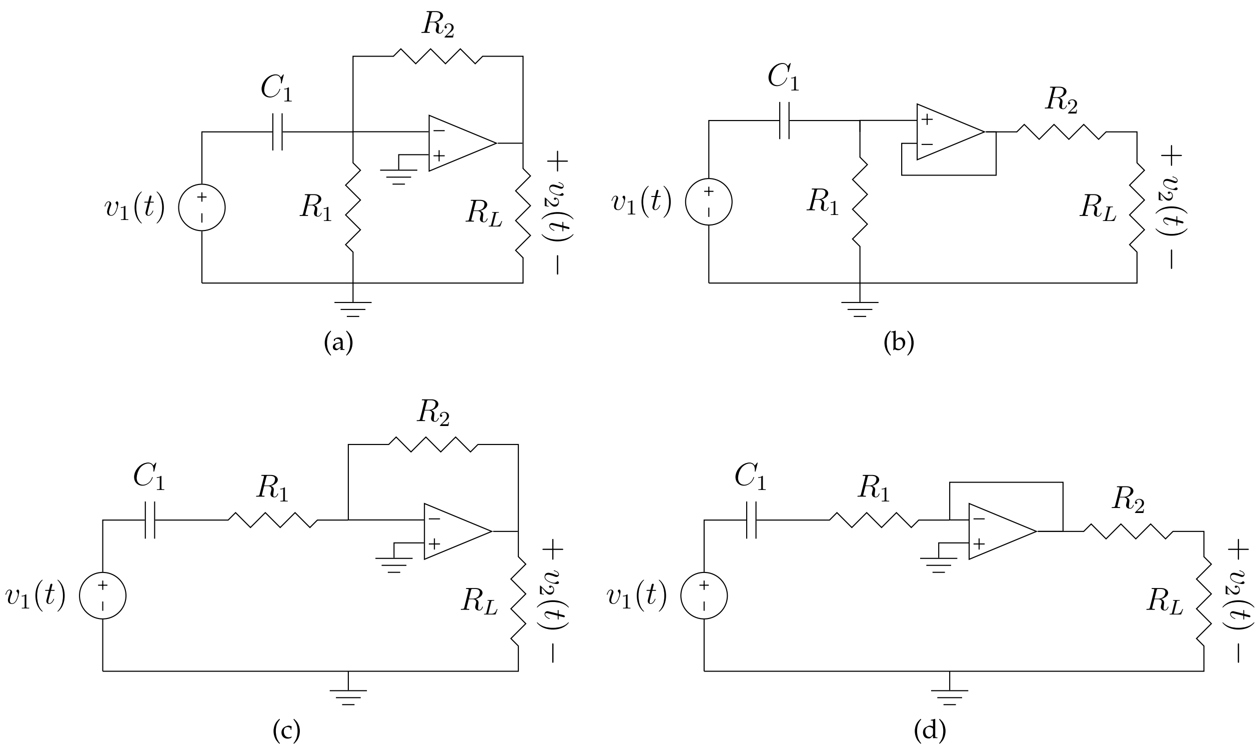


Figure 2: Circuits for Question 5

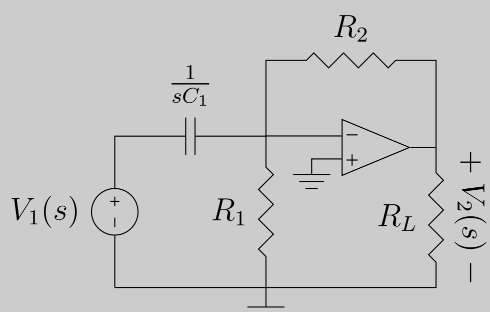
5. **OpAmp Circuit Analysis**

A student has proposed the four circuits in Fig. 2 as alternatives to the AC Coupler circuit in Fig.1.

(a) (3 points) Assume zero initial conditions, convert the circuit in Fig. 2a to the s-domain and calculate its transfer-function from $V_1(s)$ to $V_2(s)$.

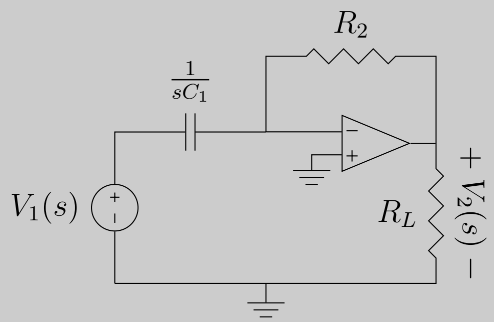
Solution:

First convert to the s-domain:



(+1 point)

Because $V_N = V_P = 0$ there is no current in R_1 so this circuit is equivalent to the standard inverting OpAmp circuit:



(+1 point)

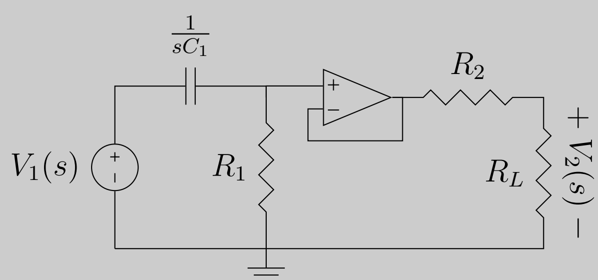
from which we can calculate the transfer-function:

$$V_2(s) = T(s)V_1(s), \qquad T(s) = -\frac{R_2}{\frac{1}{sC_1}} = -sR_2C_1. \qquad \text{(+1 point)}$$

(b) (3 points) Assume zero initial conditions, convert the circuit in Fig. 2b to the s-domain and calculate its transfer-function from $V_1(s)$ to $V_2(s)$.

Solution:

First convert to the s-domain:



(+1 point)

The first half of the circuit is the voltage divider:

$$V_P(s) = \frac{R_1}{R_1 + \frac{1}{sC_1}} V_1(s) = \frac{sR_1C_1}{1 + sR_1C_1} V_1(s) \qquad \text{(+1 point)}$$

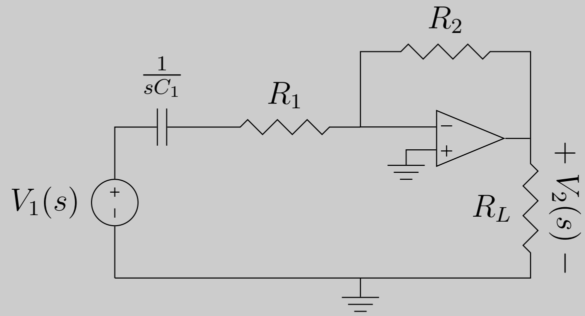
This circuit is followed by a buffer after which the transfer-function can be again calculated using voltage division:

$$V_2(s) = T(s)V_1(s), \qquad T(s) = \frac{R_L}{R_L + R_2} \frac{sR_1C_1}{1 + sR_1C_1}. \qquad \text{(+1 point)}$$

(c) (3 points) Assume zero initial conditions, convert the circuit in Fig. 2c to the s-domain and calculate its transfer-function from $V_1(s)$ to $V_2(s)$.

Solution:

First convert to the s-domain:



(+1 point)

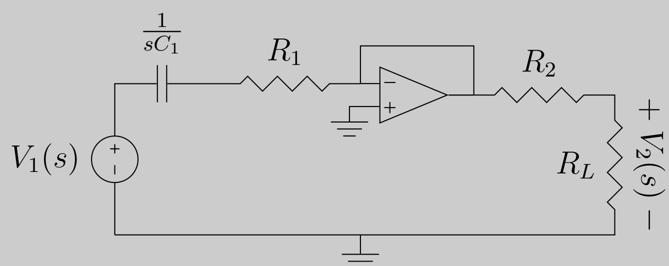
This is a standard inverting OpAmp circuit from which

$$V_2(s) = T(s)V_1(s), \qquad T(s) = -\frac{R_2}{R_1 + \frac{1}{sC_1}} = -\frac{sR_2R_1C_1}{1 + sR_1C_1} \qquad \text{(+2 points)}$$

(d) (3 points) Assume zero initial conditions, convert the circuit in Fig. 2d to the s-domain and calculate its transfer-function from $V_1(s)$ to $V_2(s)$.

Solution:

First convert to the s-domain:



(+1 point)

This is a standard inverting OpAmp circuit with a zero resistance (short-circuit) in the feedback path followed by a voltage divider from which

$$V_2(s) = T(s)V_1(s), \qquad T(s) = -\frac{R_2}{R_2 + R_l} \times \frac{0}{R_1 + \frac{1}{sC_1}} = 0 \qquad \text{(+2 points)}$$

(e) (4 points) Assume that $R_L \gg R_2$ and compare the transfer-functions you obtained with the one given in Problem 3. Which circuit in Fig. 2 is the best replacement for the AC Coupler in Fig. 1? Justify your answer.

Solution:

The transfer-function from Problem 3 is:

$$T(s) = \frac{sRC_1}{1 + sRC_1} = \frac{s}{s + \frac{1}{RC_1}}$$

which is a high-pass filter with cutoff frequency $\frac{1}{RC_1}$ and gain

$$\lim_{\omega \to \infty} |T(j\omega)| = 1. \qquad \text{(+1 point)}$$

The transfer-functions from each item in Problem 5 are:

$$T_a(s) = -sR_2C_1$$
$$T_b(s) = \frac{R_L}{R_L + R_2} \times \frac{sR_1C_1}{1 + sR_1C_1}$$
$$T_c(s) = -\frac{sR_2R_1C_1}{1 + sR_1C_1}$$
$$T_d(s) = 0$$

If $R_L \gg R_2$

$$T_a(s) = -sR_2C_1$$
$$T_b(s) = \frac{sR_1C_1}{1 + sR_1C_1}$$
$$T_c(s) = -\frac{sR_2R_1C_1}{1 + sR_1C_1}$$
$$T_d(s) = 0$$
(+1 point)

Compared to $T(s)$, (b) and (c) are the best candidates. Both as high-pass filters with cutoff frequencies $\frac{1}{R_1C_1}$. (+1 point)
However, T_b and T_c have gains

$$\lim_{\omega\to\infty} |T_c(j\omega)| = \frac{R_1C_1}{R_1C_1} = 1, \qquad \lim_{\omega\to\infty} |T_c(j\omega)| = \frac{R_2R_1C_1}{R_1C_1} = R_2$$
(+1 point)

making T_b the best replacement.

(f) (2 points) Name two advantages of the active circuit you chose in part (e) when compared with the passive circuit in Fig. 1.

Solution:

The OpAmp separates the source, $v_1(t)$, from the load R_L. Even if R_L is a short-circuit, the current provided by $v_1(t)$ will remain unaffected. (+1 point)

The cutoff frequency of the circuit in Fig. 2b is independent of the load resistance R_L. (+1 point)

(g) (2 points) Identify at least two design flaws in the circuits of Fig. 2.

Solution:

In Fig. 2a the resistor R_1 is connected between ground and the virtual ground, (v_N), thus serving no function in that circuit. (+1 point)

In Fig. 2d the feedback resistance is zero which connects the output directly to the virtual ground, (v_N), thus making the output zero no matter the input voltage, (v_1). (+1 point)