MAE 140 HW6 Solution

9.4 Find Laplace transform of $f(t) = 20\left(e^{-200t} - 2e^{-100t}\right)u(t)$. Locate the poles & zeros!

$$F(s) = 20\left(\mathcal{L}\{e^{-200t}u(t)\} - 2\mathcal{L}\{e^{-100t}u(t)\}\right) = 20\left(\frac{1}{s+200} - 2\frac{1}{s+100}\right)$$

$$= 20\left(\frac{s+100-2(s+200)}{(s+200)(s+100)}\right) = 20\left(\frac{-s-300}{(s+100)(s+200)}\right)$$

$$\hookrightarrow \boxed{F(s) = -20\frac{(s+300)}{(s+100)(s+200)}} \rightarrow \boxed{\begin{array}{l}\text{Poles}: \{-100, -200\} \\ \text{Zeros}: \{-300\}\end{array}}$$

9.9 Find Laplace of $f(t) = 10\left(3 - 5t - 2e^{-15t}\right)u(t)$. Locate poles & zeros!

$$F(s) = 10\left(\frac{3}{s} - \frac{5}{s^2} - \frac{2}{s+15}\right) = 10\left(\frac{3s(s+15) - 5(s+15) - 2s^2}{s^2(s+15)}\right)$$

$$= 10\left(\frac{s^2 - 40s - 75}{s^2(s+15)}\right) \rightarrow \boxed{F(s) = 10\frac{(s+1.7945)(s-41.7945)}{s^2(s+15)}}$$

$$\hookrightarrow \boxed{\begin{array}{l}\text{Poles}: \{0, 0, -15\} \\ \text{Zeros}: \{-1.7945, 41.7945\}\end{array}}$$

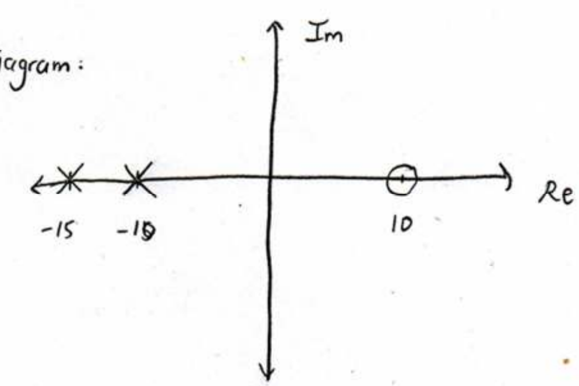
9.10 Find Laplace and plot the pole-zero diagrams.

a) $f_1(t) = \left(25e^{-15t} - 20e^{-10t}\right)u(t)$

$$F_1(s) = 25\frac{1}{s+15} - 20\frac{1}{s+10} = 5\left(\frac{5(s+10) - 4(s+15)}{(s+10)(s+15)}\right) = \boxed{5\frac{(s-10)}{(s+10)(s+15)}}$$

$$\boxed{\begin{array}{l}\text{Poles}: \{-10, -15\} \\ \text{Zeros}: \{10\}\end{array}}$$

Diagram:



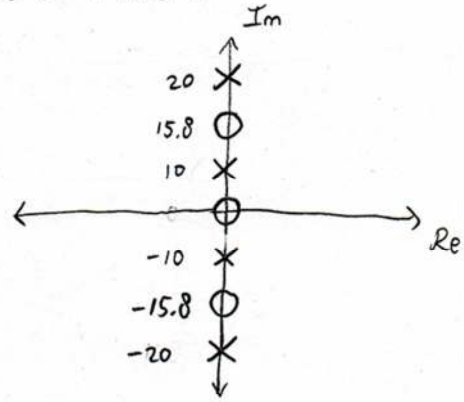
b) $f_2(t) = 10\left(\cos 10t + \cos 20t\right)u(t)$

$$F_2(s) = 10\left(\frac{s}{s^2+100} + \frac{s}{s^2+400}\right) = 10 \cdot s\left(\frac{s^2+400+s^2+100}{(s^2+100)(s^2+400)}\right)$$

$$\boxed{F_2(s) = 20\,\frac{s(s^2+250)}{(s^2+100)(s^2+400)}}$$

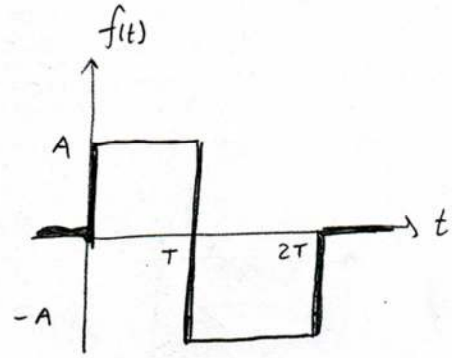
↳ Poles: $\{\pm 10j, \pm 20j\}$
Zeros: $\{\pm 15.8j, 0\}$

Diagram:



9.17

a) Write an expression for $f(t)$ using step functions.



$$\boxed{f(t) = A\,u(t) - 2A\,u(t-T) + A\,u(t-2T)}$$

or $f(t) = A\left(u(t) - 2u(t-T) + u(t-2T)\right)$

b) Use time domain translation property to find the Laplace!

$$\boxed{F(s) = A\left(\frac{1}{s} - 2\,\frac{1}{s}e^{-Ts} + \frac{1}{s}e^{-2Ts}\right)} \quad \text{or} \quad F(s) = \frac{A}{s}\left(1 - 2e^{-Ts} + e^{-2Ts}\right)$$

c) Verify the transformation by using the definition of Laplace transform.

$$F(s) = \int_{0^-}^{\infty} f(t)\,e^{-st}\,dt = \int_{0^-}^{\infty} A\left(u(t) - 2u(t-T) + u(t-2T)\right)e^{-st}\,dt$$

$$= A\left(\int_0^T e^{-st}\,dt - \int_T^{2T} e^{-st}\,dt\right) = A\left(-\frac{1}{s}\left(e^{-Ts} - e^{0}\right) + \frac{1}{s}\left(e^{-2Ts} - e^{-Ts}\right)\right)$$

$$= \frac{A}{s}\left(1 - e^{-Ts} - e^{-Ts} + e^{-2Ts}\right) \rightarrow \boxed{F(s) = \frac{A}{s}\left(1 - 2e^{-Ts} + e^{-2Ts}\right)}$$

shown!

9.20 Find the inverse Laplace!

a) $F_1(s) = \dfrac{50}{s(s+50)} = \dfrac{A}{s} + \dfrac{B}{s+50}$

use the residual method

$A = \left(\dfrac{50}{s(s+50)} \cdot s\right)\Big|_{s=0} = 1$

$B = \left(\dfrac{50}{s(s+50)} (s+50)\right)\Big|_{s=-50} = -1$

$F_1(s) = \dfrac{1}{s} - \dfrac{1}{s+50}$

$$\boxed{f_1(t) = \left(1 - e^{-50t}\right) u(t)}$$

b) $F_2(s) = \dfrac{(s+1)}{(s+2)(s+3)} = \dfrac{A}{s+2} + \dfrac{B}{s+3}$

$A = \left(\dfrac{s+1}{s+3}\right)\Big|_{s=-2} = -1$

$B = \left(\dfrac{s+1}{s+2}\right)\Big|_{s=-3} = 2$

$F_2(s) = -\dfrac{1}{s+2} + \dfrac{2}{s+3}$

$$\boxed{f_2(t) = \left(-e^{-2t} + 2e^{-3t}\right) u(t)}$$

9.21 Find inverse Laplace!

a) $F_1(s) = \dfrac{s+30}{s(s+40)} = \dfrac{A}{s} + \dfrac{B}{s+40}$

$A = \left(\dfrac{s+30}{s+40}\right)\Big|_{s=0} = \dfrac{3}{4}$

$B = \left(\dfrac{s+30}{s}\right)\Big|_{s=-40} = \dfrac{1}{4}$

$F_1(s) = \dfrac{1}{4}\left(\dfrac{3}{s} + \dfrac{1}{s+40}\right)$

$$\boxed{f_1(t) = \frac{1}{4}\left(3 + e^{-40t}\right) u(t)}$$

b) $F_2(s) = \dfrac{(s+10)(s+20)}{s(s+50)(s+100)} = \dfrac{A}{s} + \dfrac{B}{s+50} + \dfrac{C}{s+100}$

$A = \left(\dfrac{(s+10)(s+20)}{(s+50)(s+100)}\right)\Big|_{s=0} = 0.04$

$B = \left(\dfrac{(s+10)(s+20)}{s(s+100)}\right)\Big|_{s=-50} = -0.48$

$C = \left(\dfrac{(s+10)(s+20)}{s(s+50)}\right)\Big|_{s=-100} = 1.44$

$F_2(s) = \dfrac{0.04}{s} - \dfrac{0.48}{s+50} + \dfrac{1.44}{s+100}$

$$\boxed{f_2(t) = \left(0.04 - 0.48e^{-50t} + 1.44e^{-100t}\right) u(t)}$$

$\boxed{9.23}$ Find inverse Laplace!

a) $F_1(s) = \frac{900}{(s+10)^2+30^2} = 30\,\frac{30}{(s+10)^2+30^2}$ → exponentially decaying sine

$$\boxed{f_1(t) = 30\, e^{-10t} \sin(30t)\, u(t)}$$

$\frac{\omega}{(s+\alpha)^2+\omega^2} \rightarrow e^{-\alpha t}\sin(\omega t)$

b) $F_2(s) = \frac{3(s+10)}{(s+10)^2+30^2} = \frac{A(s+10)+B(30)}{(s+10)^2+30^2} \longrightarrow \begin{matrix} A=3 \\ B=0 \end{matrix}$

$F_2(s) = 3\,\frac{s+10}{(s+10)^2+30^2} \rightarrow \boxed{f_2(t) = 3\, e^{-10t}\cos(30t)\, u(t)}$

$\boxed{9.27b}$ Find inverse Laplace for $F_2(s) = \frac{5(s^2+9)}{s(s^2+25)} = \frac{5s^2+45}{s(s^2+25)}$

$F_2(s) = \frac{A}{s} + \frac{Bs+5C}{s^2+25} = \frac{A(s^2+25)+Bs^2+5Cs}{s(s^2+25)} = \frac{(A+B)s^2+5Cs+25A}{s(s^2+25)}$

$$\left.\begin{aligned} C &= 0 \\ A &= 45/25 = 1.8 \\ B &= 5-A = 3.2 \end{aligned}\right\}$$

$F_2(s) = \frac{1.8}{s} + \frac{3.2s}{s^2+25}$

$$\boxed{f_2(t) = (1.8 + 3.2\cos(5t))\, u(t)}$$

$\boxed{9.31a}$ Find inverse Laplace for $F_1(s) = \frac{(s+100)^2}{(s+50)^2(s+200)} = \frac{s^2+200s+10000}{(s+50)^2(s+200)}$

$F_1(s) = \frac{A}{s+50} + \frac{B}{(s+50)^2} + \frac{C}{s+200} = \frac{A(s+50)(s+200)+B(s+200)+C(s+50)^2}{(s+50)^2(s+200)}$

$= \frac{(A+C)s^2 + (250A+B+100C)s + (10000A+200B+2500C)}{(s+50)^2(s+200)}$

$$\begin{bmatrix} 1 & 0 & 1 \\ 250 & 1 & 100 \\ 10000 & 200 & 2500 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ 200 \\ 10000 \end{bmatrix} \rightarrow \boxed{\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{bmatrix} 0.5556 \\ 16.6667 \\ 0.4444 \end{bmatrix}}$$

$$F_1(s) = \frac{5/9}{s+50} + \frac{150/9}{(s+50)^2} + \frac{4/9}{s+200}$$

$$\boxed{f_1(t) = \frac{1}{9}\left(5e^{-50t} + 150te^{-50t} + 4e^{-200t}\right)u(t)}$$

9.33 a Find inverse Laplace for $F_1(s) = \dfrac{s(s+10)(s+20)}{(s+5)(s+30)(s+50)} = \dfrac{s^3+30s^2+200s}{(s+5)(s+30)(s+50)}$

$$F_1(s) = \frac{A}{s+5} + \frac{B}{s+30} + \frac{C}{s+50} + D$$

$D \neq 0$ because the order of the numerator is equal to the denumerator.

We can use the residual method except for D..

$$A = \left(\frac{s(s+10)(s+20)}{(s+30)(s+50)}\right)\Bigg|_{s=-5} = -\frac{1}{3}$$

$$B = \left(\frac{s(s+10)(s+20)}{(s+5)(s+50)}\right)\Bigg|_{s=-30} = 12$$

$$C = \left(\frac{s(s+10)(s+20)}{(s+5)(s+50)}\right)\Bigg|_{s=-50} = -\frac{200}{3}$$

D is the only one that can reach s^3

↳ $D = 1$

↳ $$F_1(s) = \frac{-1/3}{s+5} + \frac{12}{s+30} - \frac{200/3}{s+50} + 1$$

$$\boxed{f_1(t) = \frac{1}{3}\left(-e^{-5t} + 36e^{-30t} - 200e^{-50t}\right)u(t) + \delta(t)}$$

$\delta(t)$ is the impulse function.