

MAE140 – Linear Circuits – Fall – 2016
Final, December 6th

Instructions

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a handheld calculator with no communication capabilities.
2. You have 170 minutes.
3. Do not forget to write your name and student number.
4. This exam has 6 questions, for a total of 60 points and 2 bonus points.

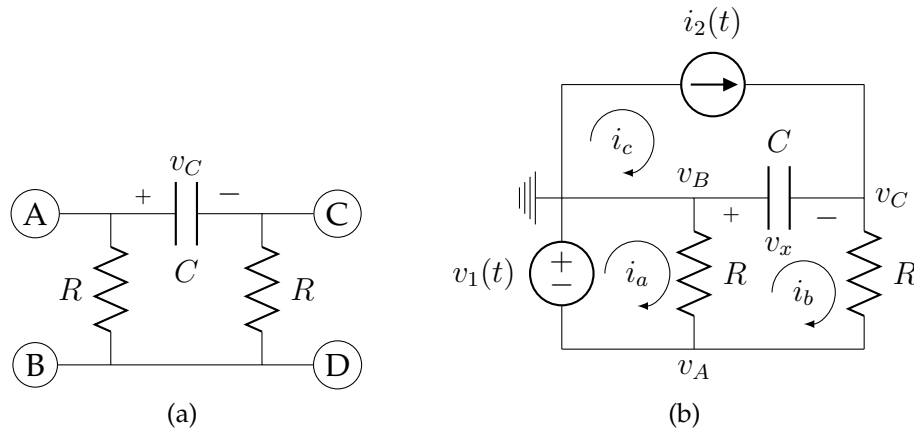


Figure 1: Circuits for Questions 1 and 2

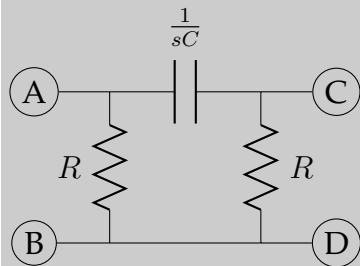
1. Circuit Equivalence

IMPORTANT: All impedances should be given as a ratio of two polynomials!

- (a) (3 points) Assume zero-initial conditions and find the s -domain equivalent circuit as seen from terminals (A) and (B) in Fig. 1a.

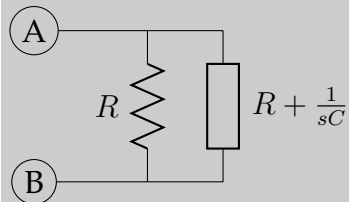
Solution:

First convert the circuit to the s -domain to obtain assuming zero initial conditions:



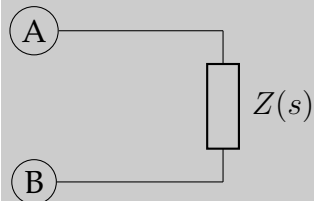
(+1 point)

As seen from (A) to (B) we associate the capacitor and resistor:



(+1 point)

which is equivalent to:



(+1/2 point)

where

$$\begin{aligned}
 Z(s) &= \frac{1}{\frac{1}{R} + \frac{1}{R + \frac{1}{sC}}} \\
 &= \frac{R}{1 + \frac{s}{s + \frac{1}{RC}}} \\
 &= \frac{R}{\frac{2s + \frac{1}{RC}}{s + \frac{1}{RC}}} \\
 &= \frac{R \left(s + \frac{1}{RC} \right)}{2s + \frac{1}{RC}} \quad (+1/2 \text{ point})
 \end{aligned}$$

- (b) (3 points) Assume zero-initial conditions and find the s -domain equivalent circuit as seen from terminals (C) and (B) in Fig. 1a.

Solution:

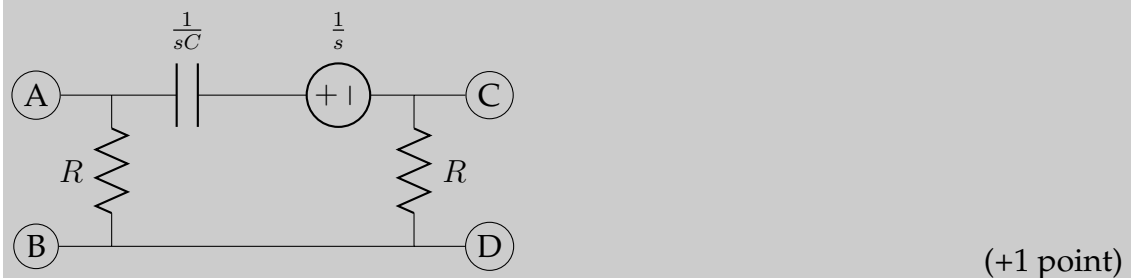
The circuit has already been converted to the s -domain in part (a).

Just notice that the circuit is the same as seen from (C) to (B) or from (A) to (B) so the answer is the same as in part (a). (+3 points)

- (c) (4 points) Assume a non-zero initial condition $v_C(0) = 1V$ as indicated and repeat parts (a) and (b).

Solution:

Start again by converting to the s -domain but this time consider a non-zero initial condition across the capacitor with polarity matching the circuit diagram:

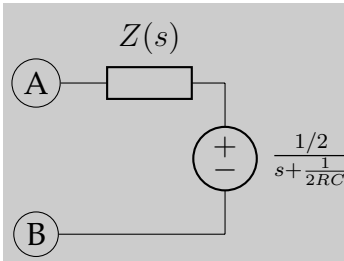


The Thevenin equivalent as seen from (A) to (B) or (C) to (B) will have the same impedances as calculate in parts (a) and (b). (+1 point)

To calculate the Thevenin voltage from (A) to (B) we can use voltage division:

$$\begin{aligned}
 V_T(s) = V_{AB}(s) &= \frac{R}{2R + \frac{1}{sC}} \frac{1}{s} \\
 &= \frac{1/2}{s + \frac{1}{2RC}} \quad (+1/2 \text{ point})
 \end{aligned}$$

to obtain the equivalent circuit:



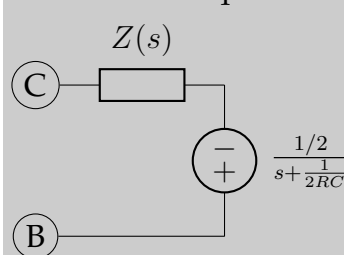
(+1/2 point)

To calculate the Thevenin voltage from (B) to (C) we note that

$$V_T(s) = V_{CB}(s) = -V_{AB}(s) = \frac{1/2}{s + \frac{1}{2RC}}$$

(+1/2 point)

to obtain the equivalent circuit:



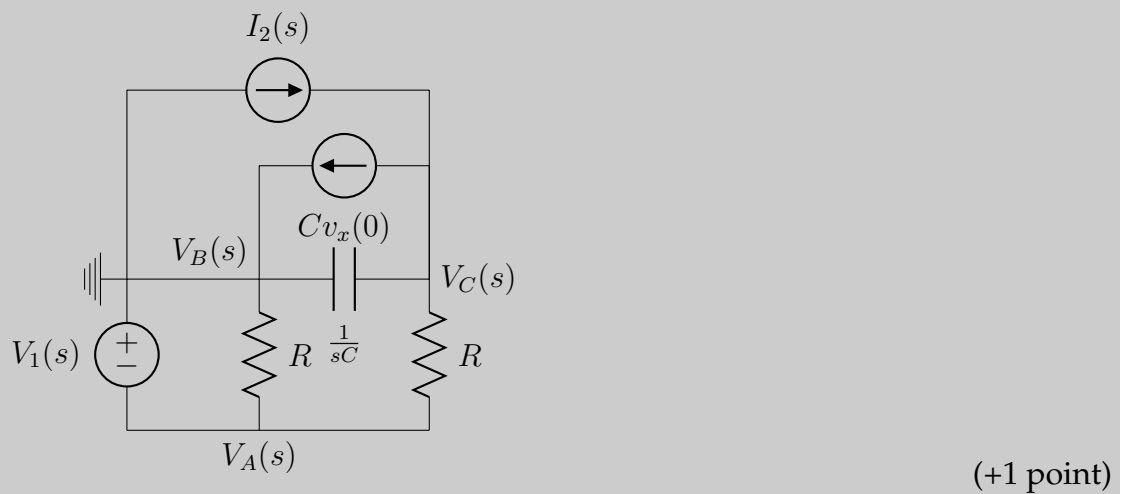
(+1/2 point)

2. Circuit Analysis

- (a) (4 points) Convert the circuit in Fig. 1b to the s -domain and formulate its node-voltage equations. Use the node-voltage and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns. Do not assume zero initial conditions. Make sure your final equations only involve node-voltages.

Solution:

With an eye on node-voltage analysis we convert to the s -domain:



Since

$$V_B(s) = 0 \quad (+1 \text{ point})$$

and $V_A(s)$ is connected to a voltage source so that

$$V_A(s) = -V_1(s) \quad (+1 \text{ point})$$

we only need to write KCL at node C , which we do by inspection:

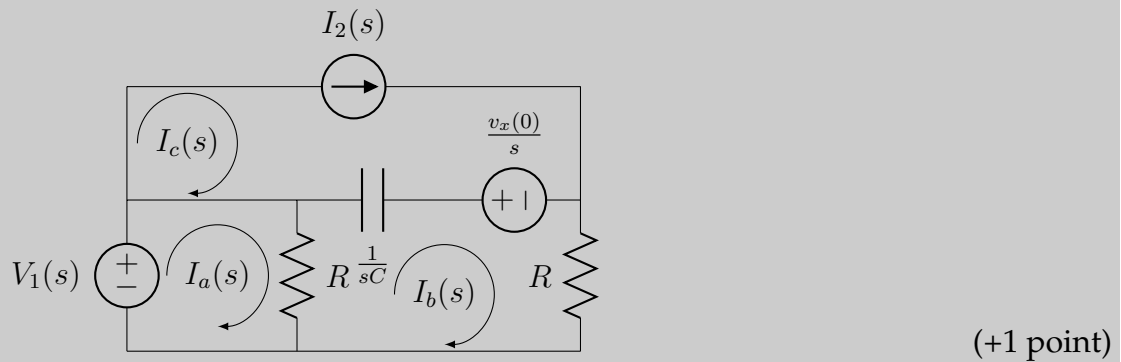
$$\left[-\frac{1}{R} \quad sC + \frac{1}{R} \right] \begin{pmatrix} V_A(s) \\ V_C(s) \end{pmatrix} = (I_2(s) - Cv_x(0)) \quad (+1 \text{ point})$$

The above equations need to be solved for the node voltages $V_A(s)$, $V_B(s)$ and $V_C(s)$.

- (b) (4 points) Convert the circuit in Fig. 1b to the s -domain and formulate its mesh-current equations. Use the mesh-currents and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns. Do not assume zero initial conditions. Make sure your final equations only involve mesh-currents.

Solution:

With an eye on mesh-current analysis we convert to the s -domain:



Since $I_2(s)$ is on an external mesh

$$I_c(s) = I_2(s) \quad (+1 \text{ point})$$

and we only need to write KVL at meshes a and b , which we do by inspection:

$$\begin{bmatrix} R & -R & 0 \\ -R & 2R + \frac{1}{sC} & -\frac{1}{sC} \end{bmatrix} \begin{pmatrix} I_a(s) \\ I_b(s) \\ I_c(s) \end{pmatrix} = \begin{pmatrix} V_1(s) \\ -\frac{v_x(0)}{s} \end{pmatrix} \quad (+2 \text{ point2})$$

The above equations need to be solved for the mesh currents $I_a(s)$, $I_b(s)$ and $I_c(s)$.

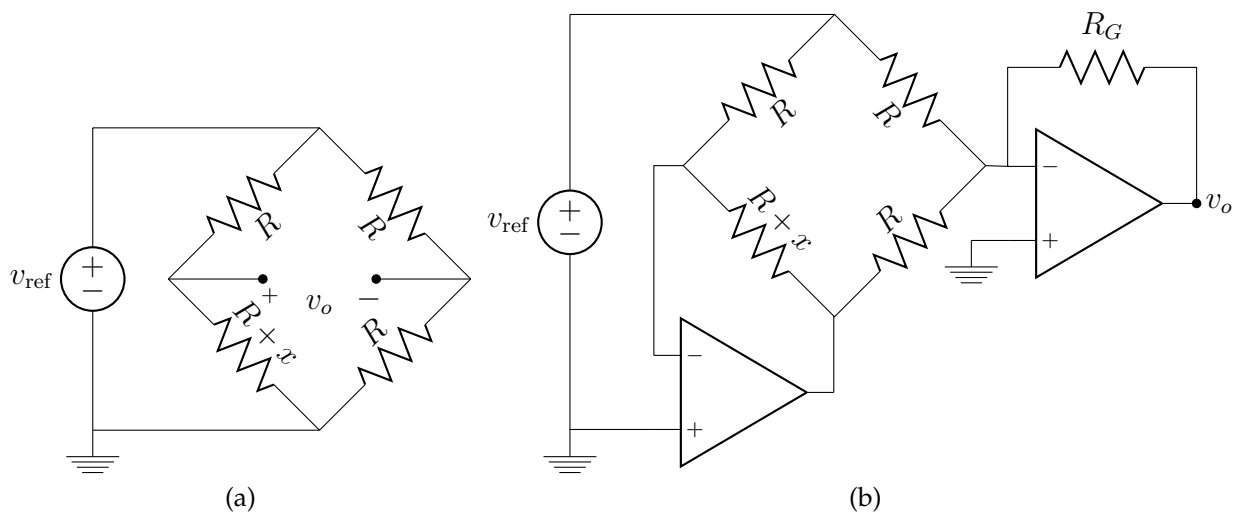


Figure 2: Circuits for Question 3

- (c) (2 points) Express the s -domain voltage $V_x(s)$ in terms of the node voltages and mesh currents from parts (a) and (b).

Solution:

From the node voltages:

$$V_x(s) = V_B(s) - V_C(s) = -V_C(s) \quad (+1 \text{ point})$$

From the mesh currents:

$$V_x(s) = \frac{I_b(s) - I_c(s)}{sC} + \frac{v_x(0)}{s} \quad (+1 \text{ point})$$

3. Wheatstone Bridge

The circuits in Fig. 2 are used with strain-gauges to measure small displacements in structures. The displacement causes a change in resistance x indicated in the figure.

(a) (5 points) For the circuit in Fig. 2a, show that

$$\frac{v_0}{v_{\text{ref}}} \approx \frac{x}{4R}$$

when x is small.

Hint: Expand in series about $x = 0$ and discard higher order terms.

Solution:

v_0^+ and v_0^- can be computed using voltage division, that is:

$$v_0^+ = \frac{R+x}{2R+x} v_{\text{ref}} = \frac{1+x/R}{2+x/R} v_{\text{ref}}, \quad (+1 \text{ point})$$

$$v_0^- = \frac{R}{2R} v_{\text{ref}} = \frac{v_{\text{ref}}}{2} \quad (+1 \text{ point})$$

so that

$$v_0 = v_0^+ - v_0^- = \frac{1+x/R}{2+x/R} v_{\text{ref}} - \frac{v_{\text{ref}}}{2} \quad (+1 \text{ point})$$

$$= \frac{2(1+x/R) - (2+x/R)}{2(2+x/R)} v_{\text{ref}} = \frac{x/R}{2(2+x/R)} v_{\text{ref}} \quad (+1 \text{ point})$$

When x/R is small

$$\frac{v_0}{v_{\text{ref}}} = \frac{x/R}{2(2+x/R)} \approx \frac{x/R}{4} \quad (+1 \text{ point})$$

This is the first-order term in the Taylor series.

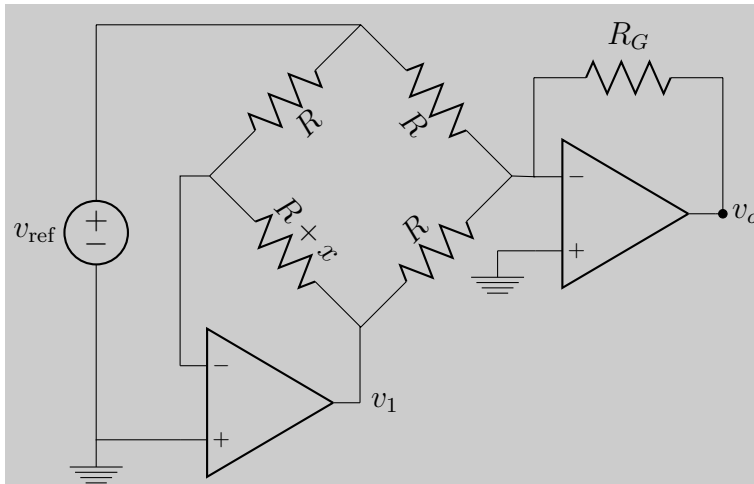
(b) (5 points) For the circuit in Fig. 2b, show that

$$\frac{v_0}{v_{\text{ref}}} = \frac{xR_G}{R^2}$$

Hint: Don't get fooled by the drawing; this is simpler than what it seems!

Solution:

The circuit consists of two inverting OpAmp circuits. For clarity label the output of the first OpAmp as in:



(+1 point)

and write for the standard inverting amplifier:

$$v_1 = -\frac{R+x}{R}v_{\text{ref}} \quad (+1 \text{ point})$$

and for the standard inverting summer:

$$v_o = -\frac{R_G}{R}v_{\text{ref}} - \frac{R_G}{R}v_1 \quad (+1 \text{ point})$$

Combining the two expressions

$$\begin{aligned} v_o &= \frac{R+x}{R} \frac{R_G}{R} v_{\text{ref}} - \frac{R_G}{R} v_{\text{ref}} \\ &= \left(\frac{R+x}{R} - 1 \right) \frac{R_G}{R} v_{\text{ref}} \\ &= \frac{R+x-R}{R} \frac{R_G}{R} v_{\text{ref}} \\ &= \frac{x}{R} \frac{R_G}{R} v_{\text{ref}} \end{aligned} \quad (+1 \text{ point})$$

from where

$$\frac{v_o}{v_{\text{ref}}} = \frac{x R_G}{R^2} \quad (+1 \text{ point})$$

(c) (1 point (bonus)) Name one advantage and one disadvantage of each circuit.

Solution:

Circuit (a): advantage: simpler; disadvantage: no way to set the gain;

Circuit (b): disadvantage: complex; advantage: gain can be set through R_G ;

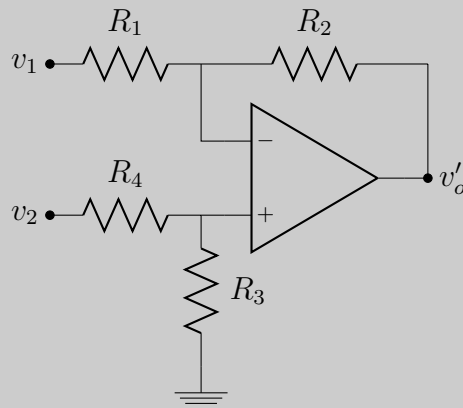
(+1 bonus point)

4. (10 points) **OpAmp Design**

Design an OpAmp circuit that can be used to amplify the voltage v_o shown in the circuit shown in Fig. 2a by a factor $K > 0$ to produce a voltage $v'_o = K v_o$. Explain how your circuit works.

Solution:

A straightforward solution is to use a subtractor:



(+1 point)

with which

$$v'_o = \frac{R_3}{R_3 + R_4} \frac{R_1 + R_2}{R_1} v_2 - \frac{R_2}{R_1} v_1 \quad (+1 \text{ point})$$

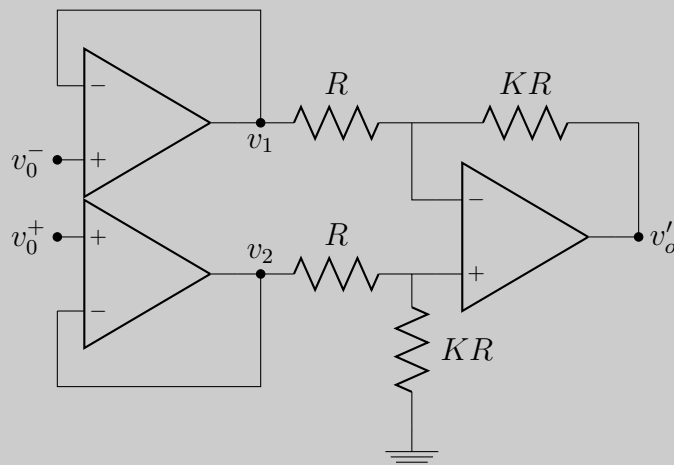
and set

$$R_1 = R, \quad R_2 = KR, \quad R_3 = KR, \quad R_4 = R \quad (+4 \text{ points})$$

so that

$$\begin{aligned} v'_o &= \frac{KR}{KR + R} \frac{R + KR}{R} v_2 - \frac{KR}{R} v_1 \\ &= K(v_2 - v_1) \end{aligned} \quad (+1 \text{ point})$$

In order to avoid loading of the bridge one cannot connect v_2 and v_1 directly to v_0^+ and v_0^- . Instead, one should use buffers. The complete circuit would be as follows:



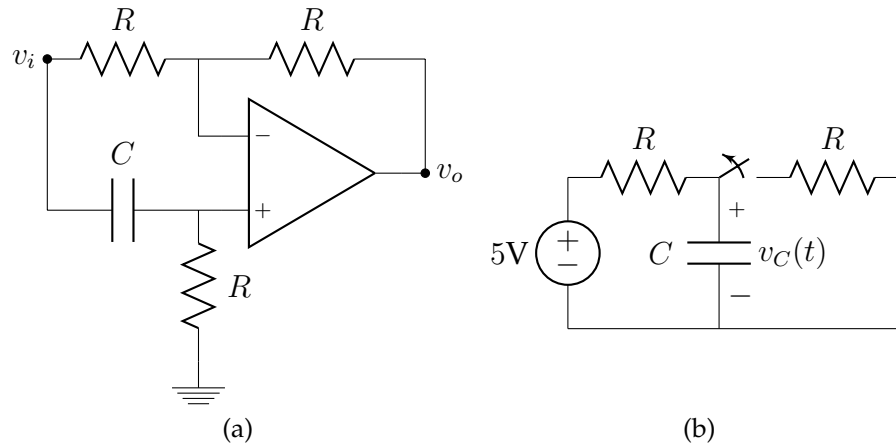


Figure 3: Circuits for Questions 5 and 6

(+3 points)

NOTE TO GRADERS: Alternative solutions are possible.

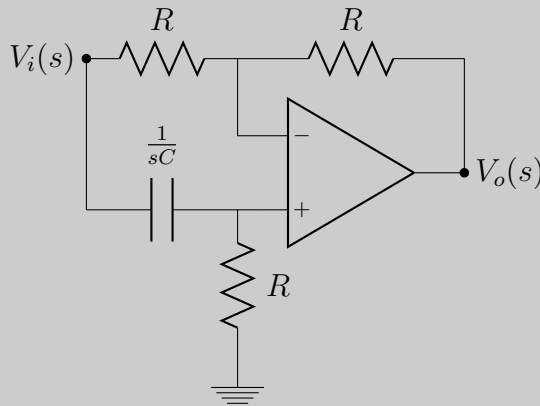
5. Frequency Response

- (a) (3 points) Show that the transfer function $T(s) = V_o(s)/V_i(s)$ for the circuit in Fig. 3a is:

$$T(s) = \frac{s - \omega_a}{s + \omega_a}, \quad \omega_a = \frac{1}{RC}$$

IMPORTANT: For the rest of this question set $\omega_a = 1\text{rad/s}$.

Solution: Assuming zero initial conditions and converting to the s -domain:



(+1 point)

The circuit is a subtractor from which:

$$V_o(s) = \frac{Z_3}{Z_3 + Z_4} \frac{Z_1 + Z_2}{Z_1} V_i(s) - \frac{Z_2}{Z_1} V_i(s) \quad (+1/2 \text{ point})$$

with

$$Z_1 = Z_2 = Z_3 = R, \quad Z_4 = \frac{1}{sC}. \quad (+1/2 \text{ point})$$

Putting it all together:

$$V_o(s) = \frac{R}{R + \frac{1}{sC}} \frac{2R}{R} V_i(s) - \frac{R}{R} V_i(s) \quad (+1/2 \text{ point})$$

$$= \left(\frac{2s}{s + \frac{1}{RC}} - 1 \right) V_i(s)$$

$$= \frac{2s - s - \frac{1}{RC}}{s + \frac{1}{RC}} V_i(s)$$

$$= \frac{s - \frac{1}{RC}}{s + \frac{1}{RC}} V_i(s) \quad (+1/2 \text{ point})$$

- (b) (3 points) Sketch the magnitude and phase of the frequency response $T(j\omega)$.

Solution:

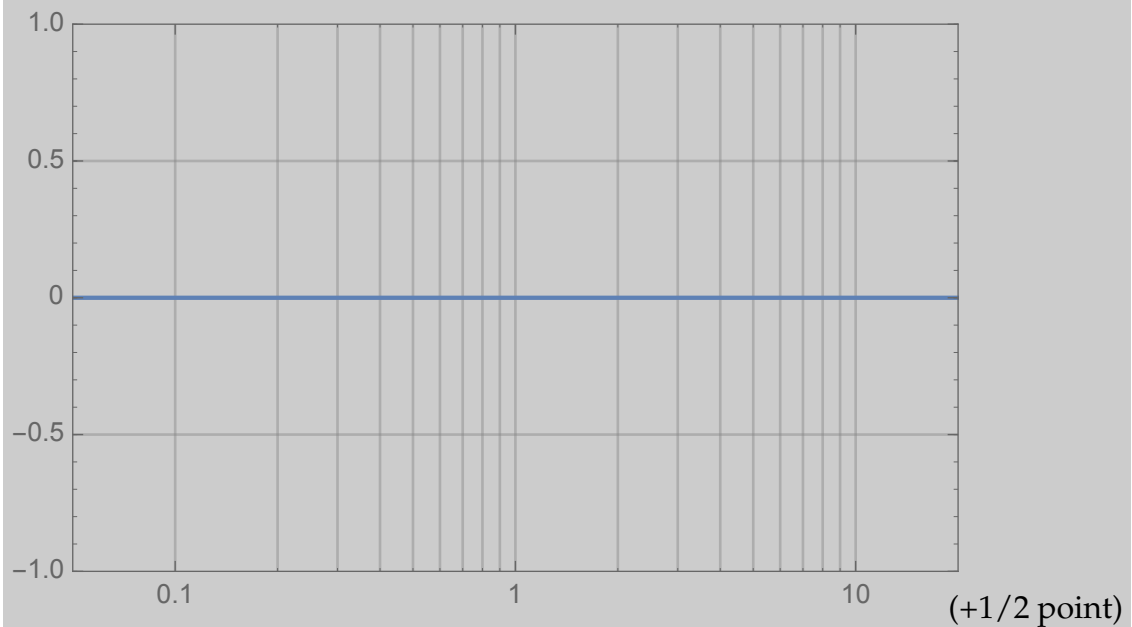
The frequency response is

$$T(j\omega) = \frac{j\omega - \omega_a}{j\omega + \omega_a} \quad (+1 \text{ point})$$

Its magnitude is

$$|T(j\omega)| = \frac{|j\omega - \omega_a|}{|j\omega + \omega_a|} = \frac{\sqrt{\omega^2 + \omega_a^2}}{\sqrt{\omega^2 + \omega_a^2}} = 1 \quad (+1/2 \text{ point})$$

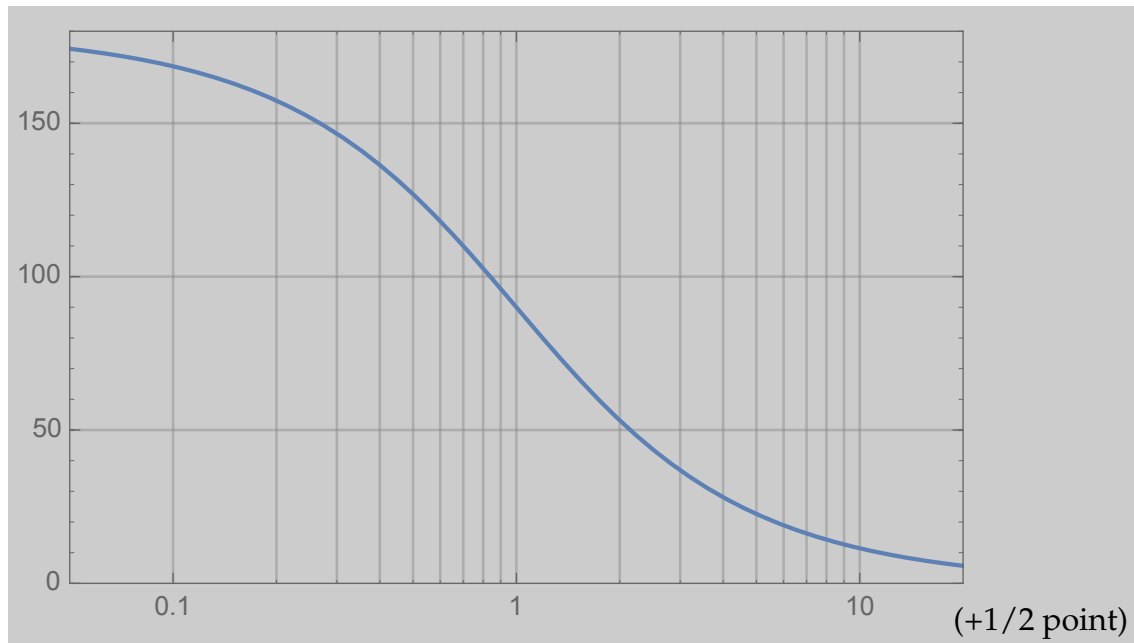
for all ω which is constant and equal to 1 for all ω . A sketch should look like:



Its phase is

$$\begin{aligned} \angle T(j\omega) &= \angle j\omega - \omega_a - \angle j\omega + \omega_a \\ &= \pi - \tan^{-1} \omega/\omega_a - \tan^{-1} \omega/\omega_a = \pi - 2 \tan^{-1} \omega/\omega_a \end{aligned} \quad (+1/2 \text{ point})$$

At $\omega = \omega_a = 1$ phase is equal to 90 degrees. A sketch should look like:



- (c) (1 point) Explain what this circuit does based on its frequency response.

Solution:

The circuit preserves the magnitude of all frequencies and shifts the phase from 180 to 0 degrees as ω grows. At $\omega = \omega_1$ the shift is equal to 90 degrees.

(+1 point)

- (d) (3 points) Use the frequency response method or a direct calculation to compute the steady-state response of the circuit to an input of the form $v_i(t) = 2 \cos(t)$ V.

Solution:

Using the frequency response method to calculate the steady state response to an input $v_i(t) = 2 \cos(\omega t)$ V we obtain

$$v_o^{SS}(t) = 2|T(j\omega)| \cos(\omega t + \angle T(j\omega)). \quad (+1 \text{ point})$$

Substituting $\omega = \omega_a = 1$ we calculate

$$T(j1) = \frac{j-1}{j+1} = j \quad (+1 \text{ point})$$

from which

$$v_o^{SS}(t) = 2 \cos(t + \pi/2) \text{ V}. \quad (+1 \text{ point})$$

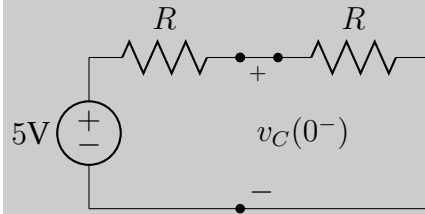
6. Circuit with Switch

The switch in the circuit in Fig. 3b is kept closed for a long time. At $t = 0$ the switch is opened.

- (a) (3 points) Determine the initial condition at the capacitor at $t = 0$ and convert the circuit to the s -domain for $t \geq 0$.

Solution:

Because the circuit has only constant sources we expect that after a long time the capacitor current will be zero, which corresponds to the following circuit:

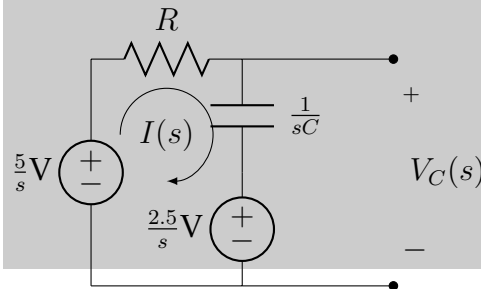


(+1 point)

so that the initial condition on the capacitor can be calculated using voltage division:

$$v_C(0^-) = \frac{R}{2R} 5V = 2.5V. \quad (+1 \text{ point})$$

The s -domain version of the circuit for $t \geq 0$ should look like:



(+1 point)

- (b) (5 points) Show that

$$v_c(t) = 5 - 2.5 e^{-t/(RC)} V, \quad t \geq 0$$

Solution:

The current in the circuit in part (a) is:

$$\begin{aligned} I(s) &= \frac{5 - 2.5}{s \left(R + \frac{1}{sC} \right)} \\ &= \frac{2.5/R}{s + \frac{1}{RC}} \end{aligned} \quad (+1 \text{ point})$$

from which

$$V_C(s) = \frac{2.5}{s} + \frac{1}{sC}I(s) \quad (+1/2 \text{ point})$$

$$= \frac{2.5}{s} + \frac{2.5 \frac{1}{RC}}{s \left(s + \frac{1}{RC} \right)}$$

$$= 2.5 \frac{s + \frac{2}{RC}}{s \left(s + \frac{1}{RC} \right)} \quad (+1/2 \text{ point})$$

Expanding in partial fractions

$$2.5 \frac{s + \frac{2}{RC}}{s \left(s + \frac{1}{RC} \right)} = \frac{\alpha}{s} + \frac{\beta}{s + \frac{1}{RC}} \quad (+1/2 \text{ point})$$

where

$$\alpha = \lim_{s \rightarrow 0} 2.5 \frac{s + \frac{2}{RC}}{s + \frac{1}{RC}} = 2.5 \frac{\frac{2}{RC}}{\frac{1}{RC}} = 5 \quad (+1/2 \text{ point})$$

$$\beta = \lim_{s \rightarrow -\frac{1}{RC}} 2.5 \frac{s + \frac{2}{RC}}{s} = 2.5 \frac{-\frac{1}{RC} + \frac{2}{RC}}{-\frac{1}{RC}} = -2.5. \quad (+1/2 \text{ point})$$

Using the Laplace inverse:

$$v_C(t) = \mathcal{L}^{-1} \left\{ \frac{\alpha}{s} + \frac{\beta}{s + \frac{1}{RC}} \right\} \quad (+1/2 \text{ point})$$

$$= \alpha + \beta e^{-t/(RC)} = 5 - 2.5 e^{-t/(RC)}, \quad t \geq 0. \quad (+1 \text{ point})$$

- (c) (2 points) Calculate the power absorbed by the resistor for $t \geq 0$.

Solution: Having calculated the voltage $v_C(t)$ we can calculate the voltage across the resistor:

$$v_R(t) = 5 - v_C(t) = 2.5 e^{-t/(RC)} \quad (+1 \text{ point})$$

from which

$$p_R(t) = \frac{v_R(t)^2}{R} = \frac{2.5^2}{R} e^{-2t/(RC)}. \quad (+1 \text{ point})$$

- (d) (1 point (bonus)) Calculate the total energy delivered by the voltage source after the switch is opened.

Solution:

From part (b), the current is

$$i(t) = \mathcal{L}^{-1}\{I(s)\} = \frac{2.5}{R} \mathcal{L}^{-1}\left\{\frac{1}{s + \frac{1}{RC}}\right\} = \frac{2.5}{R} e^{-t/(RC)}, \quad t \geq 0.$$

so that the power delivered by the source is

$$p(t) = 5i(t) = \frac{12.5}{R} e^{-t/(RC)}, \quad t \geq 0$$

and the total energy is

$$e(t) = \int_0^\infty p(\tau) d\tau = \frac{12.5}{R} \int_0^\infty e^{-\tau/(RC)} d\tau = -\frac{12.5}{R} RC e^{-\tau/(RC)} \Big|_0^\infty = 12.5 C.$$

(+1 bonus point)

Note how it does not depend on R !