MAE140 – Linear Circuits – Fall – 2017
Final, December 11th

Instructions

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a handheld calculator with no communication capabilities.
2. You have 170 minutes.
3. Do not forget to write your name and student number.
4. This exam has 6 questions, for a total of 60 points and 2 bonus points.

1. (10 points) **OpAmp Circuit Design**

   Design an OpAmp circuit that can be used to realize the following transfer-function

$$T(s) = \frac{10{,}000s}{s^2 + 10{,}010s + 100{,}000}$$

calculated between an input voltage v_i and an output voltage v_o. Show your work.

Solution:

NOTE TO GRADERS: There are many possible solutions to this problem.

First note that the roots are real, that is:

$$T(s) = \frac{10{,}000s}{s^2 + 10{,}010s + 100{,}000} = \frac{10{,}000s}{(s+10)(s+10{,}000)} \qquad \text{(+1 point)}$$

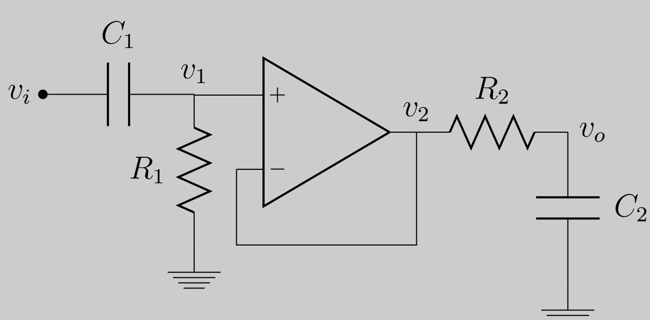
One approach is to use cascading and realize $T(s)$ as the product:

$$T(s) = T_1(s) \times T_2(s), \quad T_1(s) = \frac{s}{s+10}, \quad T_2(s) = \frac{10000}{s+10000}. \qquad \text{(+2 points)}$$

The first transfer-function is a high-pass filter with gain 1 and the second is a low-pass filter with gain 1.

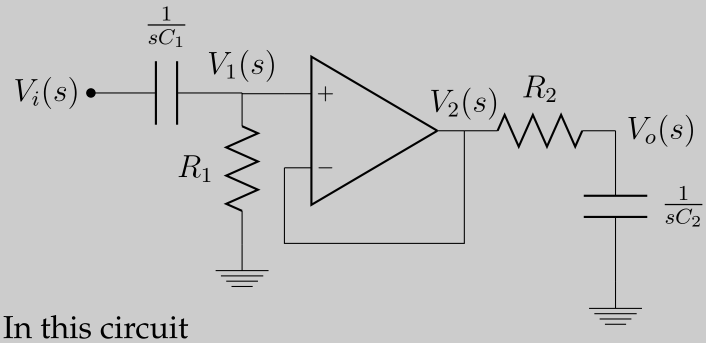
(+1 point)

One possible solution is to have two RC filters connected by a buffer as in:



(+1 point)

Converting to the s-domain:



In this circuit (+1 point)

$$V_1(s) = \frac{R_1}{R_1 + \frac{1}{sC_1}} V_i(s) = \frac{s}{s + \frac{1}{R_1C_1}} V_i(s) \qquad \text{(+1 point)}$$

and

$$V_o(s) = \frac{\frac{1}{sC_2}}{R_2 + \frac{1}{sC_2}} V_2(s) = \frac{\frac{1}{R_2C_2}}{s + \frac{1}{R_2C_2}} V_2(s) \qquad \text{(+1 point)}$$

Because of the buffer, $V_2(s) = V_1(s)$, and

$$V_o(s) = \frac{\frac{1}{R_2C_2}}{s + \frac{1}{R_2C_2}} \frac{s}{s + \frac{1}{R_1C_1}} V_i(s) \qquad \text{(+1 point)}$$

The desired transfer function can be realized by choosing R_1, C_1, R_2, and C_2 such that:

$$\frac{1}{R_1C_1} = 10, \quad \frac{1}{R_2C_2} = 10000$$

for example, $R_1 = R_2 = 10^6\Omega$, $C_1 = 10^{-7}$F, $C_2 = 10^{-10}$F. (+1 point)

2. **Frequency Response**

In this question you will analyze the circuit you designed in Question 1.

(a) (3 points) Evaluate $|T(j\omega)|$ and $\angle T(j\omega)$ for $\omega = \{0, 1000, \infty\}$rad/s.

Solution:

Note that

$$\begin{aligned} T(j\omega) &= \frac{j10000\omega}{(j\omega + 10000)(j\omega + 10)}, \\ |T(j\omega)| &= \frac{10^4|\omega|}{\sqrt{10^8 + \omega^2}\sqrt{10^2 + \omega^2}}, \\ \angle T(j\omega) &= \frac{\pi}{2} - \tan^{-1}\frac{\omega}{10000} - \tan^{-1}\frac{\omega}{10}, \quad \omega > 0, \end{aligned}$$

from which:

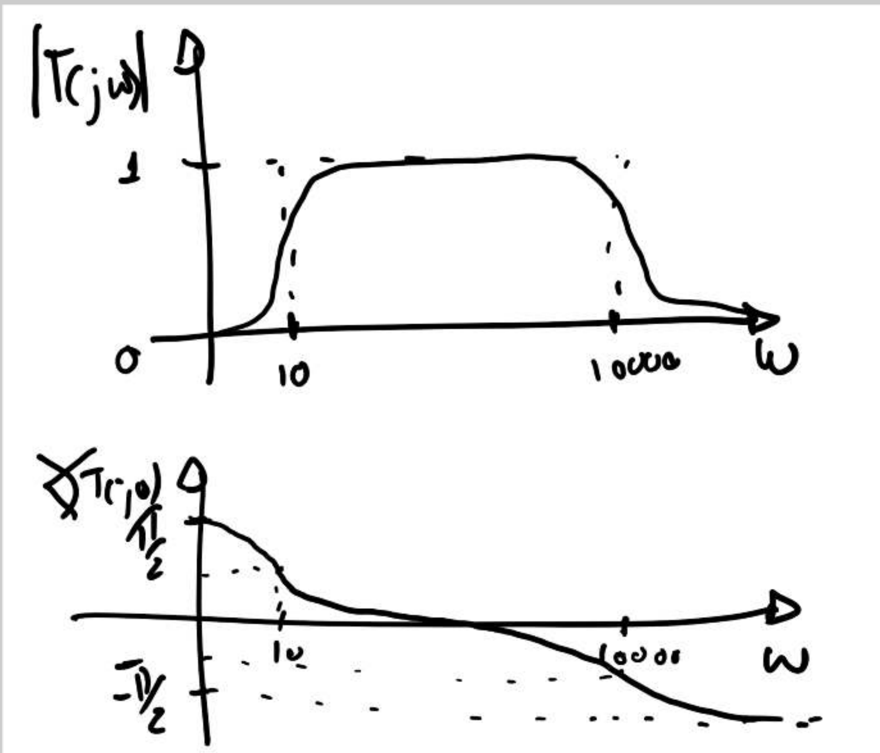
$$|T(j0)| = 0, \qquad \angle T(j0) = \frac{\pi}{2},$$

$$|T(j1000)|\frac{10^7}{\sqrt{10^8+10^6}\sqrt{10^2+10^6}} \approx 1, \quad \angle T(j0) = \frac{\pi}{2} - \tan^{-1}10^{-1} - \tan^{-1}10^{2} \approx 0,$$

$$|T(j\infty)| = 0, \qquad \angle T(j0) = -\frac{\pi}{2}.$$

(b) (3 points) Sketch the magnitude and phase of the frequency response $T(j\omega)$.

Solution:
Should look like:



(+3 points)

(c) (2 points) Classify your circuit as a filter based on its frequency response.

Solution:
Base on the above frequency response the circuit is a *bandpass filter* with a bandpass from $10 - 1000$rad/s.

(+2 point)

(d) (2 points) Use the frequency response method or a direct calculation to compute the steady-state response of the circuit to an input of the form $v_i(t) = -\cos(1000t) = \cos(\pi + 1000t)$V.

Solution:

Using the frequency response method to calculate the steady state response to an input $v_i(t) = \cos(\pi + 1000t)\text{V}$ we obtain

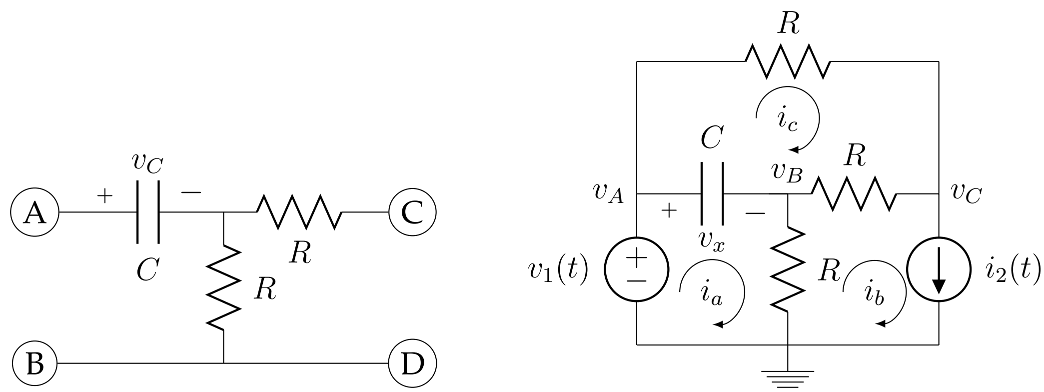
$$v_o^{SS}(t) = |T(j\omega)| \cos(\pi + \omega t + \angle T(j\omega)). \qquad \text{(+1 point)}$$

Substituting $\omega = 1000$ which we had calculated earlier

$$|T(j1000)| \approx 1, \qquad \angle T(j1000) \approx 0 \qquad \text{(+1 point)}$$

from which

$$v_o^{SS}(t) \approx \cos(\pi + 1000t)\text{V} = -\cos(1000t)\text{V}. \qquad \text{(+1 point)}$$



(a) (b)

Figure 1: Circuits for Questions 3 and 4

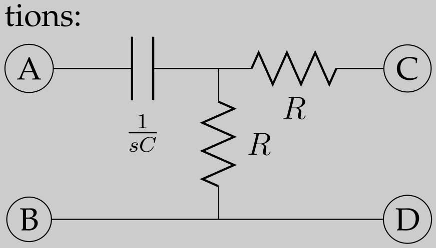
3. **Circuit Equivalence**

   IMPORTANT: All impedances should be given as a ratio of two polynomials!

   (a) (3 points) Assume zero-initial conditions and find the s-domain equivalent circuit as seen from terminals (A) and (B) in Fig. 1a.

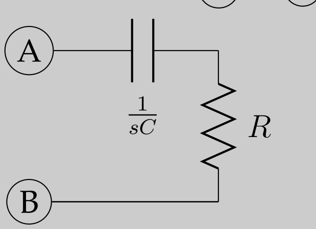
Solution:

First convert the circuit to the s-domain to obtain assuming zero initial conditions:



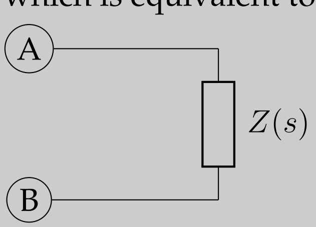
(+1 point)

As seen from (A) to (B) this circuit looks like:



(+1 point)

which is equivalent to:



(+1/2 point)

where

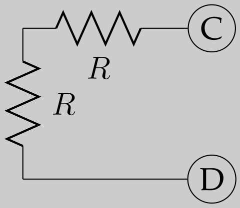
$$Z(s) = R + \frac{1}{sC} = \frac{sRC+1}{sC} \qquad \text{(+1/2 point)}$$

(b) (3 points) Assume zero-initial conditions and find the s-domain equivalent circuit as seen from terminals C and D in Fig. 1a.

Solution:
The circuit has already been converted to the s-domain in part (a).

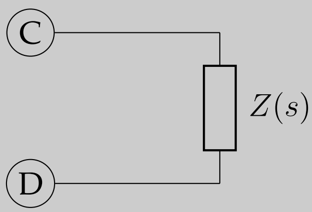
(+1 point)

As seen from C to D this circuit looks like:



(+1 point)

which is equivalent to:



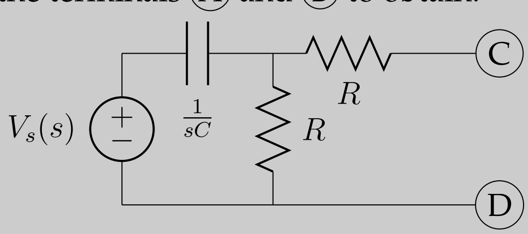
(+1/2 point)

where

$$Z(s) = R + R = 2R \qquad \text{(+1/2 point)}$$

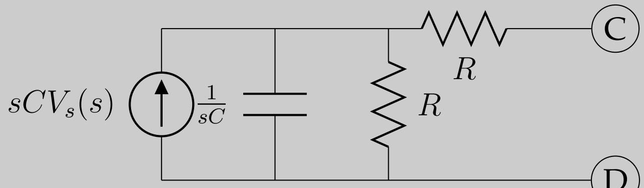
(c) (4 points) Connect a voltage source $v_s(t)$ between terminals A and B, assume zero-initial conditions and calculate the s-domain Thevenin equivalent as seen from terminals C and D.

Solution:
Start again by converting to the s-domain after connecting a source $v_1(t)$ across the terminals A and B to obtain:

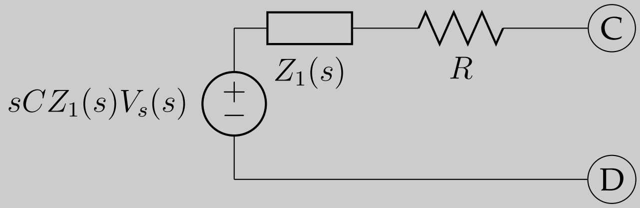


(+1 point)

We will calculate the Thevenin equivalent circuit by the following series of source transformations:



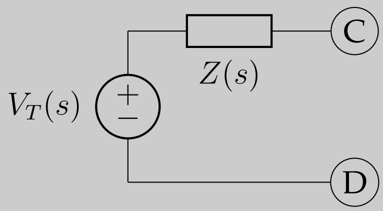
(+1 point)



where

$$Z_1(s) = \frac{1}{sC + \frac{1}{R}} = \frac{1/C}{s + \frac{1}{RC}}, \qquad \text{(+1 point)}$$

and finally:



where

$$V_T(s) = sCZ_1(s)V_s(s) = \frac{s}{s + \frac{1}{RC}} V_s(s)$$

$$Z(s) = R + Z_1(s) = R\left(1 + \frac{\frac{1}{RC}}{s + \frac{1}{RC}}\right) = R\frac{s + \frac{2}{RC}}{s + \frac{1}{RC}} \qquad \text{(+1 point)}$$

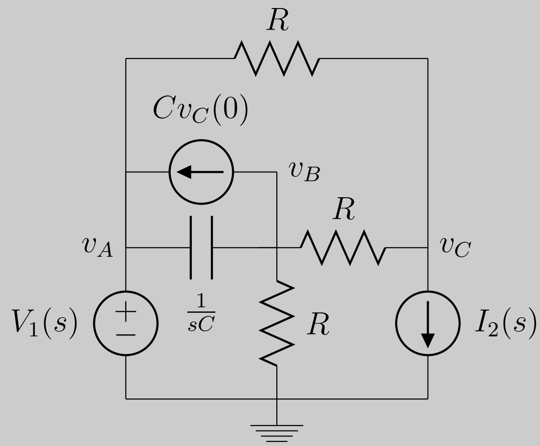
which is the Thevenin equivalent circuit.

4. **Circuit Analysis**

(a) (4 points) Convert the circuit in Fig. 1b to the s-domain and formulate its node-voltage equations. Use the node-voltage and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns. Do not assume zero initial conditions. Make sure your final equations only involve node-voltages.

Solution:

With an eye on node-voltage analysis we convert to the s-domain:



(+1 point)

We write the node-voltage equations by inspection ommiting node A:

$$\begin{bmatrix} -sC & \frac{2}{R} + sC & -\frac{1}{R} \\ 0 & -\frac{1}{R} & \frac{2}{R} \end{bmatrix} \begin{pmatrix} v_A(s) \\ v_B(s) \\ v_C(s) \end{pmatrix} = \begin{pmatrix} -Cv_C(0) \\ -I_2(s) \end{pmatrix} \qquad \text{(+1 point)}$$

along with the additional relation:

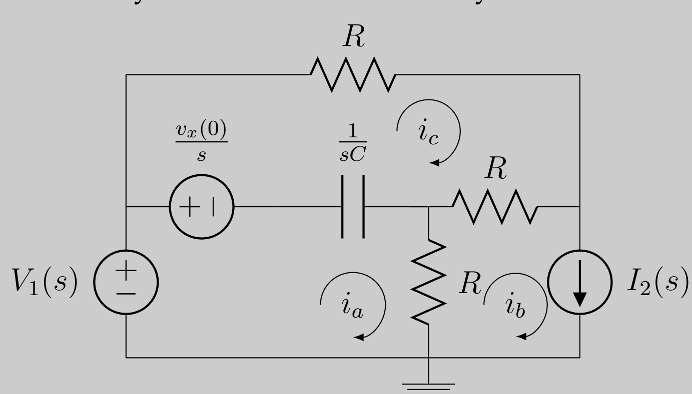
$$V_A(s) = V_1(s) \qquad \text{(+1 point)}$$

The above equations need to be solved for the node voltages $V_A(s)$, $V_B(s)$ and $V_C(s)$. (+1 point)

(b) (4 points) Convert the circuit in Fig. 1b to the s-domain and formulate its mesh-current equations. Use the mesh-currents and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns. Do not assume zero initial conditions. Make sure your final equations only involve mesh-currents.

Solution:

With an eye on mesh-current analysis we convert to the s-domain:



(+1 point)

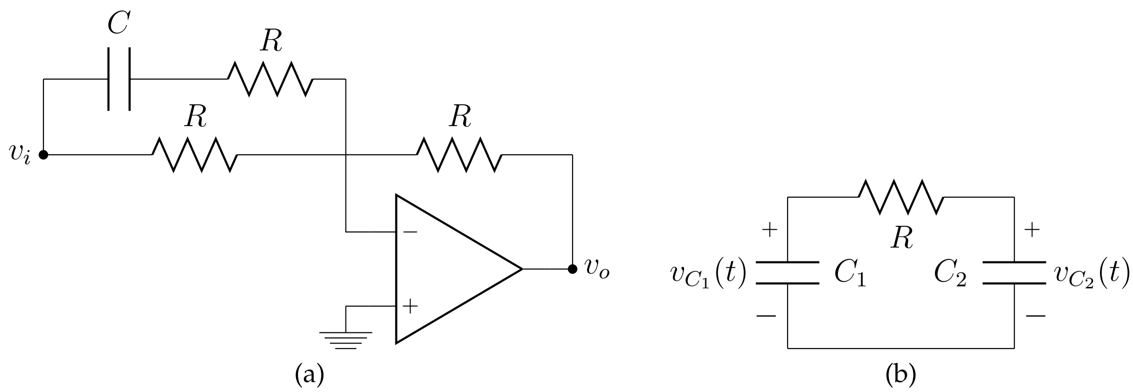


Figure 2: Circuits for Questions 5 and 6

We write the mesh-current equations by inspection ommiting mesh b:

$$\begin{bmatrix} \frac{1}{sC}+R & -R & -\frac{1}{sC} \\ -\frac{1}{sC} & -R & \frac{1}{sC}+2R \end{bmatrix} \begin{pmatrix} I_a(s) \\ I_b(s) \\ I_c(s) \end{pmatrix} = \begin{pmatrix} V_1(s) - \frac{v_x(0)}{s} \\ \frac{v_x(0)}{s} \end{pmatrix} \qquad \text{(+1 point)}$$

along with the additional relation:

$$I_b(s) = I_2(s) \qquad \text{(+1 point)}$$

The above equations need to be solved for the mesh currents $I_a(s)$, $I_b(s)$ and $I_c(s)$. (+1 point)

(c) (2 points) Express the s-domain voltage $V_x(s)$ in terms of the node voltages and mesh currents from parts (a) and (b).

Solution:
From the node voltages:

$$V_x(s) = V_A(s) - V_B(s) \qquad \text{(+1 point)}$$

From the mesh currents:

$$V_x(s) = \frac{I_a(s) - I_c(s)}{sC} + \frac{v_x(0)}{s} \qquad \text{(+1 point)}$$

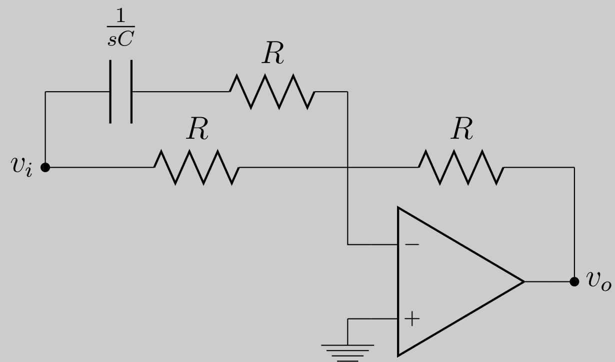
5. **OpAmp Circuit Analysis**

A student claims that the circuit in Fig. 2a is an amplifier that has a higher gain at DC than it has at higher frequencies. Answer the following questions to prove or disprove this claim.

(a) (3 points) Assume zero-initial conditions and calculate the transfer-function from the input v_i to the output v_o.

Solution:

Convert to the s-domain:



(+1 point)

and calculate:

$$
\begin{aligned}
V_o(s) &= -\frac{R}{\frac{1}{\frac{1}{R}+\frac{1}{R+\frac{1}{sC}}}}V_i(s) \\
&= -\frac{R}{\frac{1}{\frac{1}{R}\left(1+\frac{s}{s+\frac{1}{RC}}\right)}}V_i(s) \\
&= -\frac{R}{\frac{R}{\frac{2s+\frac{1}{RC}}{s+\frac{1}{RC}}}}V_i(s) \\
&= -\frac{2s+\frac{1}{RC}}{s+\frac{1}{RC}}V_i(s)
\end{aligned}
$$

(+1 point)

from which the transfer-function is:

$$T(s) = -\frac{2s+\frac{1}{RC}}{s+\frac{1}{RC}}$$

(+1 point)

(b) (2 points) Calculate the DC gain and the gain at $\omega \to \infty$.

Solution:

Evaluating

$$T(j\omega) = -\frac{2j\omega + \frac{1}{RC}}{j\omega + \frac{1}{RC}}$$

we calculate

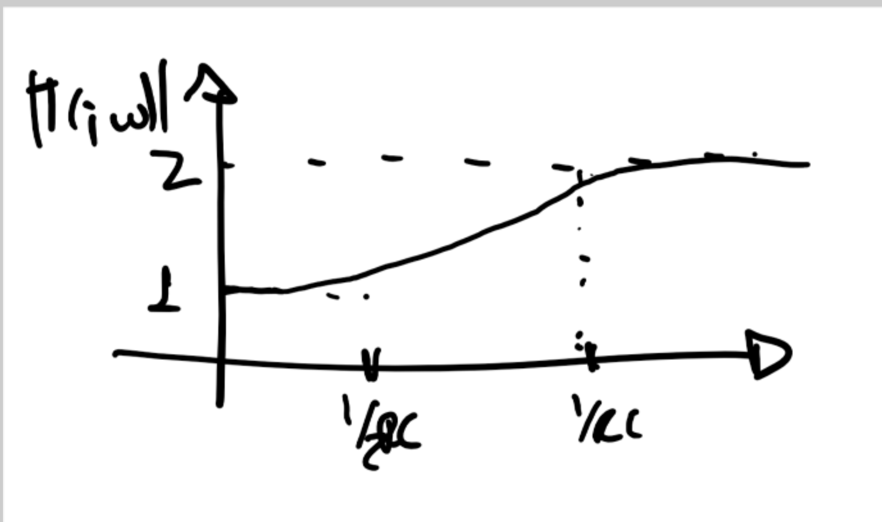
$$|T(j0)| = 1 \qquad \text{(+1 point)}$$

and

$$|T(j\infty)| = 2 \qquad \text{(+1 point)}$$

(c) (3 points) Sketch the magnitude of the frequency response of the transfer-function.

Solution:
Should look like:



(+3 points)

NOTE TO GRADERS: If the shape of the curve is correct assign all marks even if the pole and zero were not labeled correctly.

(d) (2 points) Is the student right or wrong? Explain.

Solution:
The student is wrong. The gain is twice as high at higher frequencies than it is at DC. (+2 points)

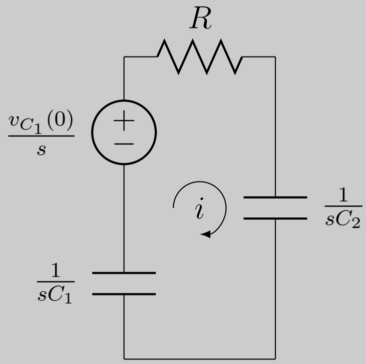
6. **Transient Analysis**

In the circuit in Fig. 2b the initial voltages are $v_{C_1}(0) = 5\text{V}$ and $v_{C_2}(0) = 0\text{V}$.

(a) (1 point) Convert the circuit to the s-domain.

Solution:

Taking into account the initial conditions:



(+1 point)

(b) (6 points) Use the Laplace transform to show that when $C_1 = C_2 = C$ then

$$v_{C_1}(t) = 2.5 + 2.5\, e^{-2t/(RC)}\mathrm{V}, \quad t \geq 0$$

and

$$v_{C_2}(t) = 2.5 - 2.5\, e^{-2t/(RC)}\mathrm{V}, \quad t \geq 0$$

Solution:

Assuming $C_1 = C_2 = C$ and using mesh analysis:

$$\left(\frac{2}{sC} + R\right) I(s) = \frac{v_{C_1}(0)}{s} = \frac{5}{s}$$

so that the current in the circuit is:

$$I(s) = \frac{5}{s\left(R + \frac{2}{sC}\right)} = \frac{5/R}{s + \frac{2}{RC}} \qquad \text{(+1 point)}$$

from which

$$V_{C_2}(s) = \frac{1}{sC} I(s) = \frac{5\frac{1}{RC}}{s\left(s + \frac{2}{RC}\right)} \qquad \text{(+1 point)}$$

and

$$V_{C_1}(s) = \frac{5}{s} - \frac{1}{sC} I(s) = \frac{5}{s} - \frac{5\frac{1}{RC}}{s\left(s + \frac{2}{RC}\right)}$$

$$= 5\frac{s + \frac{1}{RC}}{s\left(s + \frac{2}{RC}\right)} \qquad \text{(+1 point)}$$

Expanding in partial fractions

$$\frac{5\frac{1}{RC}}{s\left(s + \frac{2}{RC}\right)} = \frac{\alpha}{s} + \frac{\beta}{s + \frac{2}{RC}}$$

where

$$\alpha = \lim_{s\to 0} \frac{\frac{5}{RC}}{s+\frac{2}{RC}} = \frac{\frac{5}{RC}}{\frac{2}{RC}} = 2.5 \qquad \text{(+1/2 point)}$$

$$\beta = \lim_{s\to -\frac{2}{RC}} \frac{\frac{5}{RC}}{s} = \frac{\frac{5}{RC}}{-\frac{2}{RC}} = -2.5. \qquad \text{(+1/2 point)}$$

and

$$5\frac{s+\frac{1}{RC}}{s\left(s+\frac{2}{RC}\right)} = \frac{\gamma}{s} + \frac{\delta}{s+\frac{2}{RC}}$$

where

$$\gamma = \lim_{s\to 0} 5\frac{s+\frac{1}{RC}}{s+\frac{2}{RC}} = 5\frac{\frac{1}{RC}}{\frac{2}{RC}} = 2.5 \qquad \text{(+1/2 point)}$$

$$\delta = \lim_{s\to -\frac{2}{RC}} 5\frac{s+\frac{1}{RC}}{s} = 5\frac{-\frac{2}{RC}+\frac{1}{RC}}{-\frac{2}{RC}} = 2.5. \qquad \text{(+1/2 point)}$$

Using the Laplace inverse:

$$\begin{aligned} v_{C_2}(t) &= \mathcal{L}^{-1}\left\{\frac{\alpha}{s} + \frac{\beta}{s+\frac{2}{RC}}\right\} \\ &= \alpha + \beta e^{-2t/(RC)} = 2.5 - 2.5\,e^{-2t/(RC)}, \quad t \geq 0. \end{aligned} \qquad \text{(+1/2 point)}$$

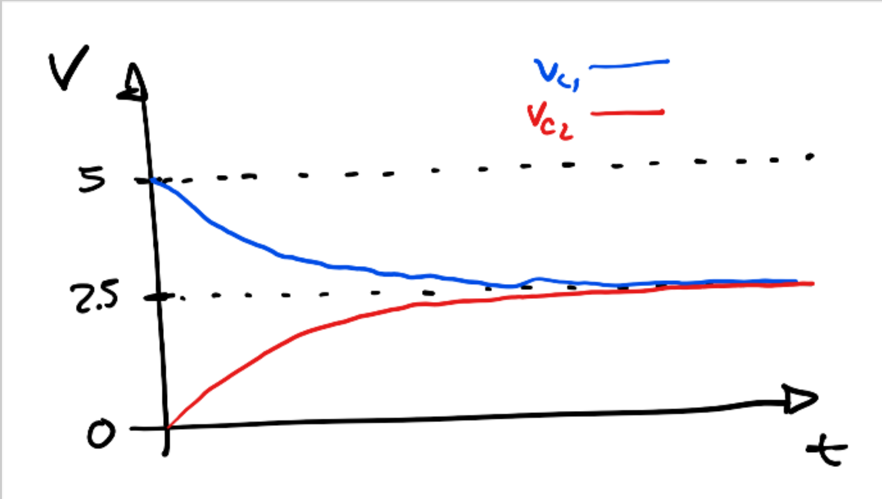
and

$$\begin{aligned} v_{C_1}(t) &= \mathcal{L}^{-1}\left\{\frac{\gamma}{s} + \frac{\delta}{s+\frac{2}{RC}}\right\} \\ &= \alpha + \beta e^{-2t/(RC)} = 2.5 + 2.5\,e^{-2t/(RC)}, \quad t \geq 0. \end{aligned} \qquad \text{(+1/2 point)}$$

(c) (3 points) Plot the voltages $v_{C_1}(t)$ and $v_{C_2}(t)$ and calculate the power absorbed by the resistor for $t \geq 0$.

Solution:

A plot of the voltages is as follows:



(+2 points)

Having calculated the voltage $v_{C_1}(t)$ and $v_{C_2}(t)$ we can calculate the voltage across the resistor:

$$v_R(t) = v_{C_1}(t) - v_{C_2}(t) = 5\, e^{-2t/(RC)}$$

from which

$$p_R(t) = \frac{v_R(t)^2}{R} = \frac{5^2}{R}\, e^{-4t/(RC)}.$$ (+1 point)

(d) (2 points (bonus)) Explain the values of the voltages v_{C_1} and v_{C_2} at $t \to \infty$ based on what you know about the relationship between charge and the voltage in capacitors. Without performing any circuit calculations, predict what would happen if $C_1 = C$ and $C_2 = C/2$?

Solution:

The initial charge on the circuit is

$$Q = C_1 v_{C_1}(0) = 5C.$$

This charge is preserved but the circuit will distribute it across the two capacitors:

$$5C = Q = C_1 v_{C_1}(\infty) + C_2 v_{C_2}(\infty)$$

Because at equilibrium there should be no current through the capacitors nor the resistor:

$$v_{C_2}(\infty) = v_{C_1}(\infty) = V_\infty.$$

When $C_1 = C_2 = C$:

$$(C + C)V_\infty = 5C$$

and

$$V_\infty = 5/2 = 2.5V$$

When $C_1 = C$ and $C_2 = C/2$:

$$(C + C/2)V_\infty = 5C$$

or

$$V_\infty = 5/(3/2) = (2/3)5V$$

(+2 bonus points)