MAE140 – Linear Circuits – Fall – 2018
Final, December 11th

Instructions

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a handheld calculator with no communication capabilities.
2. You have 170 minutes.
3. Do not forget to write your name and student number.
4. This exam has 6 questions, for a total of 60 points and 2 bonus points.

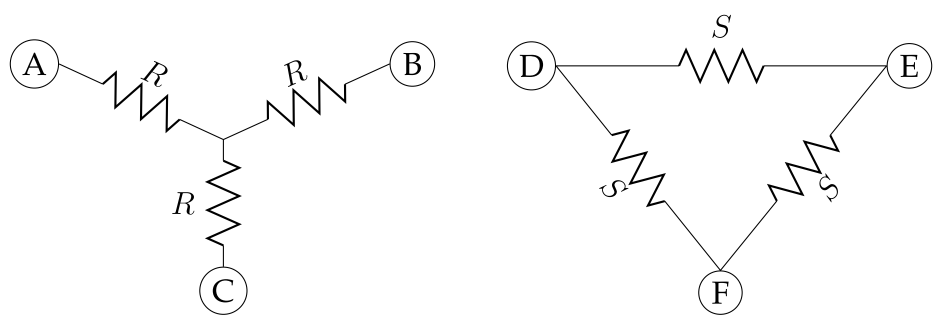


Figure 1: Circuits for Question 1

1. **Circuit Equivalence**

   Consider the circuits in Fig. 1 and answer the following questions:

   (a) (3 points) Calculate the equivalent resistance as seen from (A) and (B), (A) and (C), and (B) and (C).

   Solution: From (A) to (B), (A) to (C), and (B) to (C), the equivalent resistances are all equal to

   $$R_{AB} = R_{BC} = R_{AC} = R + R = 2R \qquad \text{(+3 points)}$$

   (b) (3 points) Calculate the equivalent resistance as seen from (D) and (E), (D) and (F), and (E) and (F).

   Solution: From (D) to (E), (D) to (F), and (E) to (F), the equivalent resistances are all equal to S in parallel with $2S$, that is

   $$R_{DE} = R_{EF} = R_{DF} = \frac{1}{\frac{1}{S} + \frac{1}{2S}} = \frac{2S}{3} \qquad \text{(+3 points)}$$

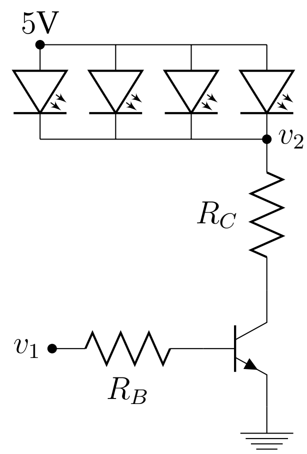


Figure 2: Circuit for Question 2

(c) (3 points) Calculate a formula for S as a function of R so that both circuits are completely equivalent.

Solution: For the circuits to be equivalent

$$R_{AB} = R_{BC} = R_{AC} = R_{DE} = R_{EF} = R_{DF} \qquad \text{(+1 point)}$$

that is

$$2R = \frac{2}{3}S \qquad \text{(+1 point)}$$

that is

$$S = 3R \qquad \text{(+1 point)}$$

(d) (1 point) Explain what would change in the formula you obtained in part (c) if the resistances R and S were replaced by impedances in the s-domain.

Solution: Nothing would change, the formula would remain the same with the resistances replaced by impedances. (+1 point)

NOTE TO GRADERS: Some students might bring up potential initial conditions and that is fine.

2. **LED Driver**

A student needs to drive four LEDs using the digital output of a microcontroller. Each LED is rated at 1.2 V and 30 mA. Because the microcontroller output pin can output a maximum of 20mA, the student needs to construct the circuit in Fig. 2 to drive the required 4×30 mA. Answer the following questions to help him or her select appropriate values for the resistors R_B and R_C.

(a) (4 points) Assume that the BJT transistor is saturated and calculate the value of R_C so that each LED has a voltage drop of 1.2 V with a current of 30 mA. Calculate the corresponding value of v_2 and the power absorbed by the resistor R_C.

Solution:

If the BJT transistor is saturated $V_{CE} \approx 0$. (+1 point)

With a voltage drop of 1.2 V, $v_2 = 5 - 1.2 = 3.8$ V (+1 point)

For a total current of $i_c = 4 \times 30 = 120$ mA the resistance R_C needs to be equal to

$$R_c = \frac{v_2}{i_c} = \frac{3.8}{0.12} \approx 31.7\Omega \qquad \text{(+1 point)}$$

The corresponding power absorbed by the resistor is

$$P_{R_C} = 3.8 \times 0.12 = 456\text{mW} \qquad \text{(+1 point)}$$

(b) (4 points) Let $v_\gamma = 0.6$ V, a transistor current gain $\beta = 100$, and calculate the value of R_B so that when $v_1 = 5$ V and each LED is operating at their rated voltage and current the transistor is saturated. Calculate the current through R_B and the power it absorbs.

Solution:

For the transistor to be saturated, $v_{CE} \approx 0$, which will happen when $i_c = 120$ mA, as calculated in part (a). (+1 point)

In this case

$$i_b = \frac{i_c}{\beta} = 1.2\text{mA} \qquad \text{(+1 point)}$$

For a $v_\gamma = 0.6$ V and when $v_1 = 5$V this current will be produced if

$$R_B = \frac{v_1 - v_\gamma}{i_b} = \frac{5 - 0.6}{1.2 \times 10^{-3}} \approx 3.7\text{K}\Omega \qquad \text{(+1 point)}$$

The corresponding power absorbed by the resistor is

$$P_{R_B} = (5 - 0.6) \times 1.2 \times 10^{-3} = 5.28\text{mW} \qquad \text{(+1 point)}$$

(c) (2 points) Explain how the circuit you helped design works as v_1 takes the values $v_1 = 0$ V and $v_1 = 5$ V. Does it keep the microcontroller output pin current below

20 mA?

Solution:

When $v_1 = 0$ V the transistor is *cut*, and therefore no current flows through the LEDs. They remain *off*. When $v_1 = 5$ V the transistor is saturated and the LEDs are on operating at their rated voltage and current for the values of R_B and R_C designed in parts (a) and (b). (+1 point)

The maximum current provided by the microcontoller pin in $i_b = 1.2$m A when $v_1 = 5$V, which is well below the rated 20 mA. (+1 point)

3. **OpAmp Design**

In this question you will design an opAmp circuit to realize the transfer-function:

$$T(s) = \frac{10(s + 1{,}000)}{s^2 + 10{,}010s + 100{,}000}$$

(a) (3 points) Calculate positive real numbers α and β so that

$$T(s) = \frac{\alpha}{s + 10} + \frac{\beta}{s + 10{,}000}$$

Solution:

The above factorization is an expansion of $T(s)$ in partial-fractions with $(s + 10)$ and $(s + 10000)$ being the factors of the polynomial $s^2 + 10010s + 100000 = (s + 10)(s + 10000)$. (+1 point)

The values of α and β are therefore:

$$\alpha = \lim_{s \to -10} (s + 10)T(s) = \lim_{s \to -10} \frac{10(s + 1000)}{s + 10000} = \frac{990}{999} = \frac{110}{111} \approx 1.0 \qquad \text{(+1 point)}$$

and

$$\alpha = \lim_{s \to -10000} (s + 1000)T(s) = \lim_{s \to -10000} \frac{10(s + 1000)}{s + 10} = \frac{9000}{999} = \frac{1000}{111} \approx 9.0$$

(+1 point)

(b) (7 points) Design a circuit using opAmps that implements $T(s)$ as the sum of the two transfer-functions from part (a).

Solution:

NOTE TO GRADERS: There are many possible solutions to this problem.

NOTE TO GRADERS: Students get no credit if they do not attempt to realize the form worked out in (a).

NOTE TO GRADERS: Students get full credit on this part if they do in terms of α and β without having calculated them in part (a).

One solution is an inverter summer followed by a gain of -1. (+1 point)

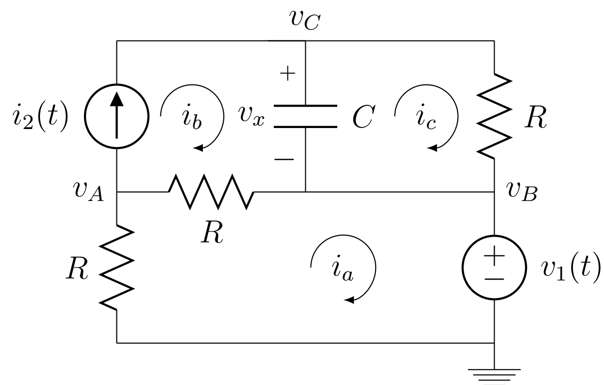
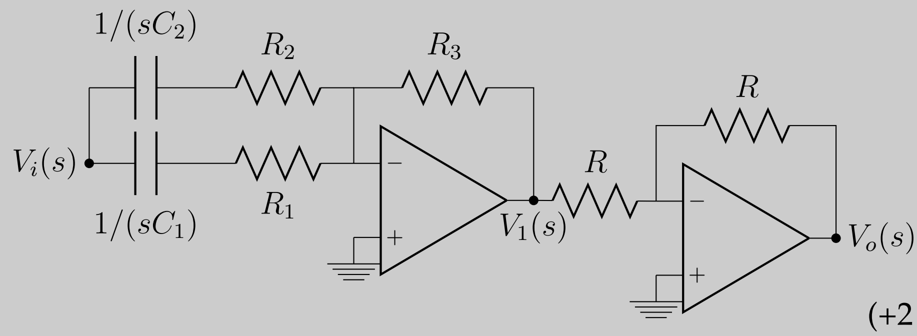


Figure 3: Circuit for Question 4

Each term is a low pass filter and one can realize them with an inverter summer as in the following circuit, already in the s-domain:



(+2 points)

In this circuit, the transfer-function from V_i to V_o is given by

$$\frac{V_1(s)}{V_i(s)} = \frac{R_3}{R_1 + \frac{1}{sC_1}} + \frac{R_3}{R_2 + \frac{1}{sC_2}} = \frac{R_3/R_1}{s + \frac{1}{R_1C_1}} + \frac{R_3/R_2}{s + \frac{1}{R_2C_2}} \quad \text{(+2 points)}$$

By setting

$$\frac{R_3}{R_1} = \alpha = 1, \qquad \frac{R_3}{R_2} = \beta = 9, \qquad \frac{1}{R_2C_2} = 10, \qquad \frac{1}{R_1C_1} = 10000 \quad \text{(+1 point)}$$

which is always, for example, by letting $R = R_3 = 900\text{K}$ from which

$$R_1 = 900\text{K}\Omega, R_2 = 100\text{K}\Omega, \quad C_2 = \frac{1}{10R_2} = 10^{-6}\text{F}, C_3 = \frac{1}{10^4 R_1} \approx 110 \times 10^{-12}\text{F}$$

(+1 point)

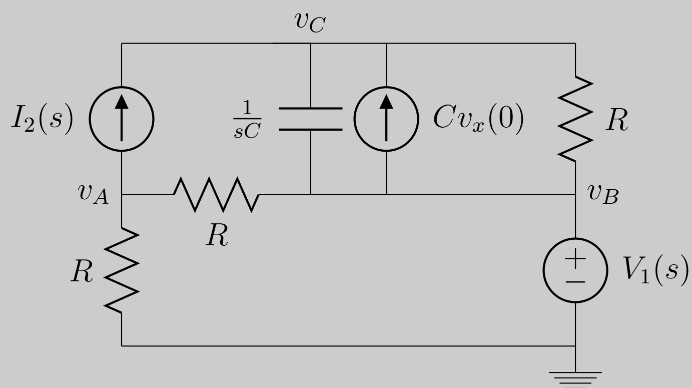
4. **Circuit Analysis**

Let $v_x(0) = 1$ V and answer the following questions:

(a) (4 points) Convert the circuit in Fig. 3 to the s-domain and formulate its node-voltage equations. Use the node-voltage and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns. Make sure your final equations only involve node-voltages.

Solution:

With an eye on node-voltage analysis we convert to the s-domain:



(+1 point)

We write the node-voltage equations by inspection omitting node B:

$$\begin{bmatrix} \frac{2}{R} & -\frac{1}{R} & 0 \\ 0 & -\frac{1}{R} - sC & \frac{1}{R} + sC \end{bmatrix} \begin{pmatrix} v_A(s) \\ v_B(s) \\ v_C(s) \end{pmatrix} = \begin{pmatrix} -I_2(s) \\ I_2(s) + Cv_x(0) \end{pmatrix}$$ (+1 point)

along with the additional relation:

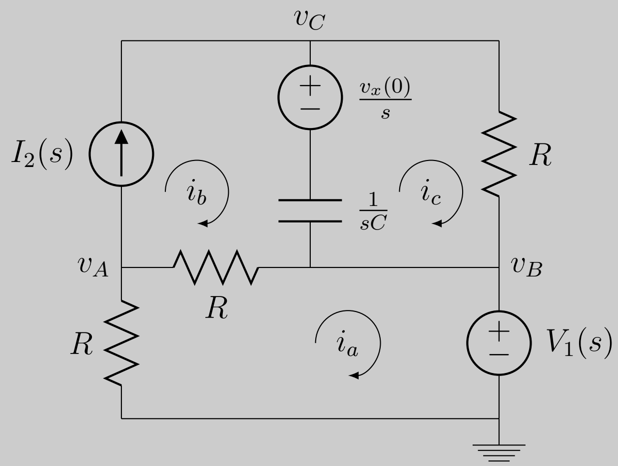
$$V_B(s) = V_1(s)$$ (+1 point)

The above equations need to be solved for the node voltages $V_A(s)$, $V_B(s)$ and $V_C(s)$. (+1 point)

(b) (4 points) Convert the circuit in Fig. 3 to the s-domain and formulate its mesh-current equations. Use the mesh-currents and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns. Make sure your final equations only involve mesh-currents.

Solution:

With an eye on mesh-current analysis we convert to the s-domain:



(+1 point)

We write the mesh-current equations by inspection omitting mesh b:

$$\begin{bmatrix} 2R & -R & 0 \\ 0 & -\frac{1}{sC} & R+\frac{1}{sC} \end{bmatrix} \begin{pmatrix} I_a(s) \\ I_b(s) \\ I_c(s) \end{pmatrix} = \begin{pmatrix} -V_1(s) \\ \frac{v_x(0)}{s} \end{pmatrix} \qquad \text{(+1 point)}$$

along with the additional relation:

$$I_b(s) = I_2(s) \qquad \text{(+1 point)}$$

The above equations need to be solved for the mesh currents $I_a(s)$, $I_b(s)$ and $I_c(s)$. (+1 point)

(c) (2 points) Write s-domain expressions for the current through the voltage source v_1 and the voltage across the current source i_2 in terms of the node voltages and mesh currents from parts (a) and (b).

Solution:

NOTE TO GRADERS: Students are free to pick their polarities and current directions but they have to be consistent.

The voltage across the source $I_2(s)$, labeled $V_2(s)$, is given in terms of the node voltages by

$$V_2(s) = V_C(s) - V_A(s) \qquad \text{(+1/2 point)}$$

or by

$$V_2(s) = \frac{v_x(0)}{s} + \frac{1}{sC}I_b(s) + R(I_b(s) - I_a(s)) \qquad \text{(+1/2 point)}$$

in terms of the mesh currents.

The current through the source $V_1(s)$, labeled $I_1(s)$, is given in terms of the mesh currents by

$$I_1(s) = -I_a(s) \qquad \text{(+1/2 point)}$$

or by

$$I_1(s) = \frac{V_B(s) - V_A(s)}{R} + \left(sC + \frac{1}{R}\right)(V_B(s) - V_C(s)) + Cv_x(0) \quad \text{(+1/2 point)}$$

in terms of the node voltages.

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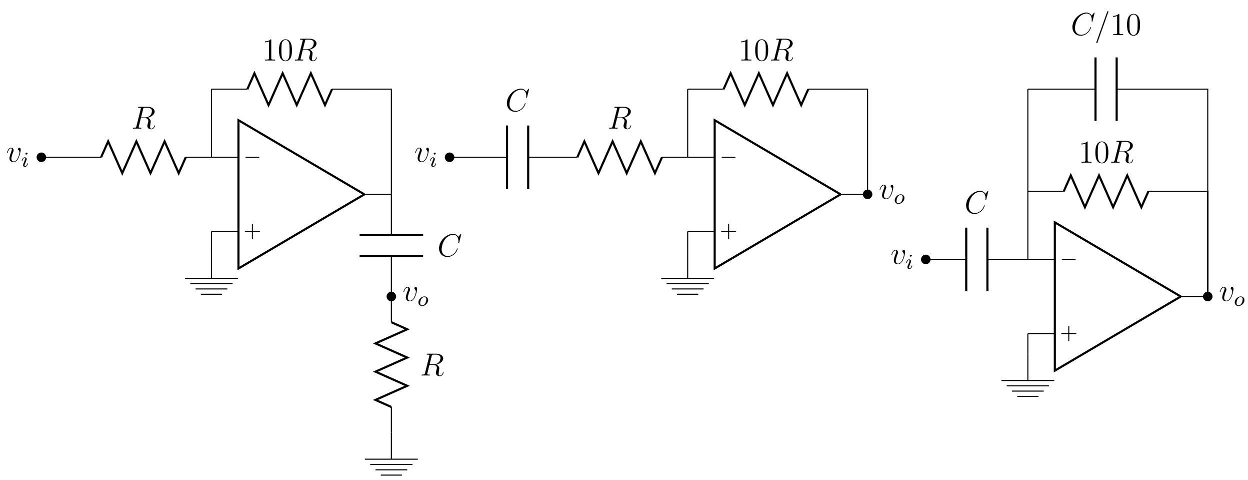


Figure 4: Circuits for Questions (5) and (6)

5. **OpAmp Circuit Analysis**

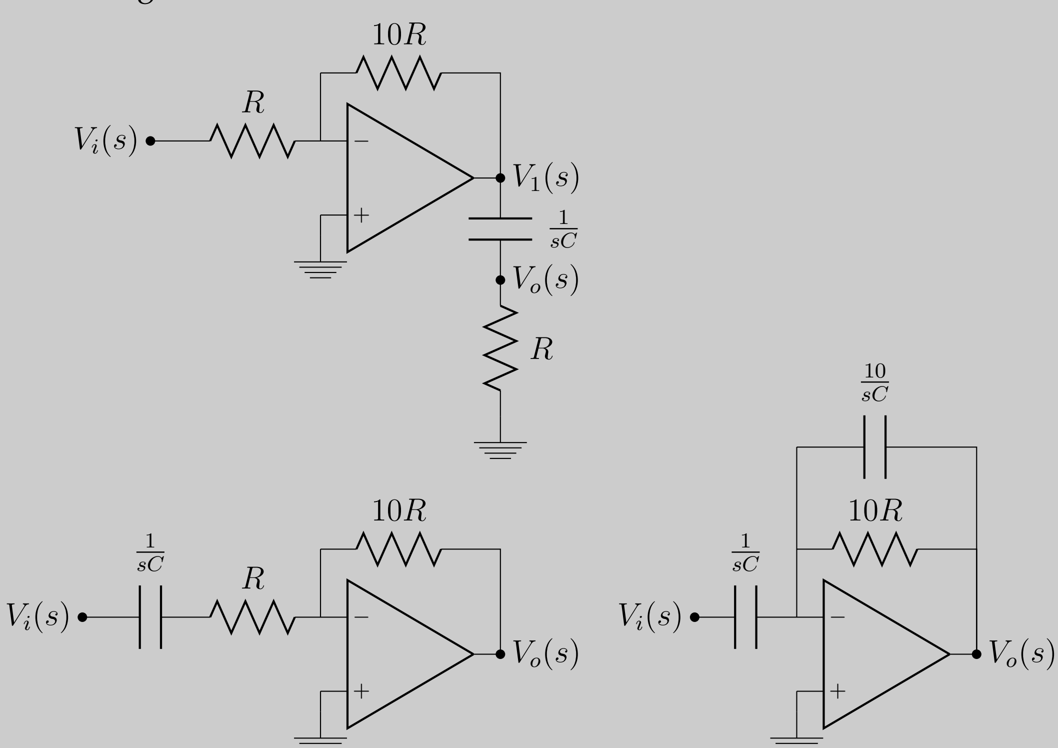
Assume zero initial-conditions for the circuits in Fig. 4 and answer the questions:

(a) (7 points) Convert the three circuits into the s-domain and show that they have identical transfer-functions

$$\frac{V_o(s)}{V_i(s)} = \frac{-10s}{s + \frac{1}{RC}}.$$

Solution:

Converting into the s-domain under zero-initial conditions.



(+1 point)

These are all based on the inverter opAmp circuits.
The transfer-function for the first circuit is given by a gain followed by a voltage divider

$$Z_1(s) = 10R, \qquad Z_2(s) = R, \qquad K(s) = \frac{R}{R + 1/(sC)} = \frac{s}{s + 1/(RC)}$$
$$T(s) = -K(s)\frac{Z_1(s)}{Z_2(s)} = \frac{-10s}{s + 1/(RC)} \qquad \text{(+2 points)}$$

The transfer-function for the second circuit is given by

$$Z_1(s) = 10R, \qquad Z_2(s) = R + \frac{1}{sC} = \frac{R(s + 1/(RC))}{s},$$
$$T(s) = -\frac{Z_1(s)}{Z_2(s)} = \frac{-10Rs}{R(s + 1/(RC)} = \frac{-10s}{s + 1/(RC)} \qquad \text{(+2 points)}$$

The transfer-function for the third circuit is given by

$$Z_1(s) = \frac{1}{\frac{1}{10R} + \frac{sC}{10}} = \frac{10/C}{s + 1/(RC)}, \qquad Z_2(s) = \frac{1}{sC},$$
$$T(s) = -\frac{Z_1(s)}{Z_2(s)} = -\frac{\frac{10/C}{s+1/(RC)}}{\frac{1}{sC}} = \frac{-10s}{s + 1/(RC)} \qquad \text{(+2 points)}$$

They are all identical, as stated.

(b) (3 points) Let $v_i(t) = V_s u(t)$ and calculate the response $v_o(t)$.

Solution:
Under zero-initial conditions, the response of all circuits is identical and is given by

$$V_o(s) = \frac{-10s}{s + \frac{1}{RC}} V_i(s) = \frac{-10s}{s + \frac{1}{RC}} \frac{V_s}{s} = \frac{-10V_s}{s + \frac{1}{RC}} \qquad \text{(+2 points)}$$

Using the Laplace inverse:

$$v_o(t) = -10V_s \mathcal{L}^{-1}\left\{\frac{1}{s + \frac{1}{RC}}\right\} = -10V_s e^{-\frac{t}{RC}} u(t). \qquad \text{(+1 point)}$$

6. **Frequency Response**

Consider the circuits in Fig. 4 and answer the following questions:

(a) (3 points) Calculate the magnitude and phase of the circuits' frequency response at $\omega = 0$, $\omega = (RC)^{-1}$, and $\omega \to \infty$.

Solution:
The frequency-response function is

$$T(j\omega) = \frac{-10j\omega}{j\omega + \frac{1}{RC}} \qquad \text{(+1/2 point)}$$

from which

$$|T(j\omega)| = \frac{10|\omega|}{\sqrt{\omega^2 + \frac{1}{(RC)^2}}} \qquad \text{(+1/2 point)}$$

and

$$\angle T(j\omega) = -\pi/2 - \tan^{-1}\omega RC, \quad \omega > 0 \qquad \text{(+1/2 point)}$$

Calculating at $\omega = 0$

$$|T(j0)| = 0 \qquad \angle T(j0) = -\pi/2 \qquad \text{(+1/2 point)}$$

At $\omega = (RC)^{-1}$

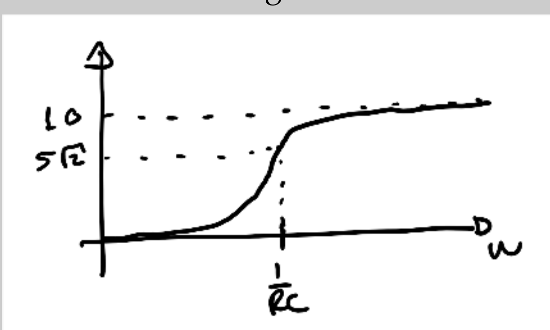
$$|T(j(RC)^{-1})| = \frac{10}{\sqrt{2}} = 5\sqrt{2} \qquad \angle T(j(RC)^{-1}) = -3\pi/4 \qquad \text{(+1/2 point)}$$

At $\omega \to \infty$

$$|T(j\infty)| = 10 \qquad \angle T(j\infty) = -\pi \qquad \text{(+1/2 point)}$$

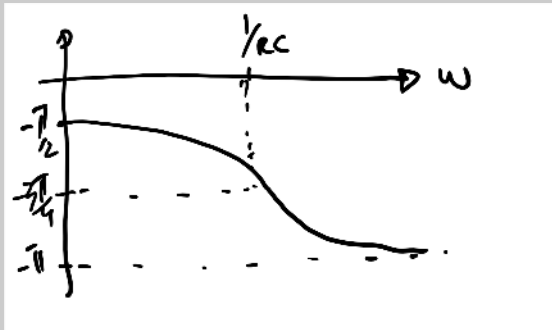
(b) (4 points) Sketch the magnitude and phase of the circuits' frequency response. What kind of filter is the circuit?

Solution:
A sketch of the magnitude is as in the following plot:



(+2 points)

A sketch of the phase is as in the following plot:



(+1 point)

This circuit is a *high-pass* filter. (+1 point)

(c) (3 points) Calculate the steady-state response $v_o^{\text{SS}}(t)$ produced in response to an input voltage $v_i(t) = 1 + \cos((RC)^{-1}t)$.

Solution:

The steady-state response can be calculated using the frequency response method and linearity. (+1 point)

That is

$$\begin{aligned} v_o^{SS}(t) &= |T(j0)|\cos(\angle T(j0)) + |T(j(RC)^{-1})|\cos((RC)^{-1}t + \angle T(j(RC)^{-1})) \\ &= 5\sqrt{2}\cos((RC)^{-1}t - 3\pi/4) \end{aligned}$$

(+2 point)

(d) (2 points (bonus)) Which of the circuits has a problem with saturation? Explain.

Solution: The output of the first circuit might become saturated if the signal v_i has a non-zero DC component, such as in part (c). The other two circuits do not suffer from this problem.

(+2 bonus points)

Have a great Winter break!