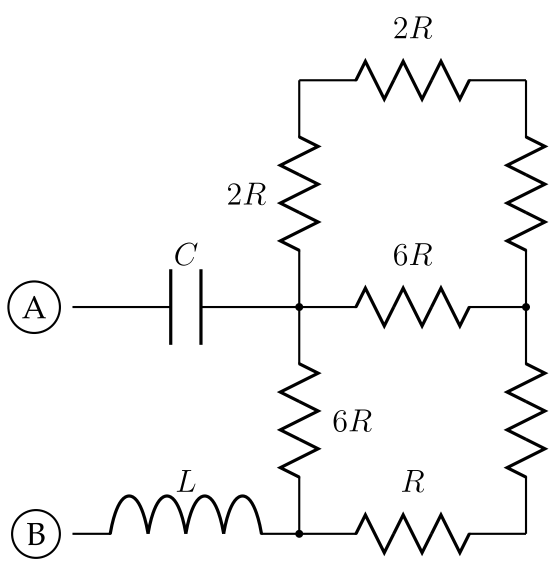
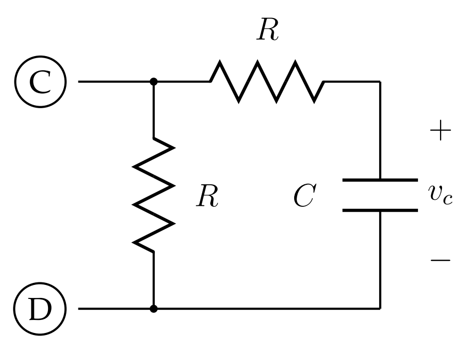
**MAE140 – Linear Circuits – Summer Session II – 2011**
**Final, September 3rd**

## Instructions

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.
2. You have 170 minutes.
3. Do not forget to write your name and student number.
4. This exam has 5 questions, for a total of 54 points and 6 bonus points.




(a) Circuit for part (a)



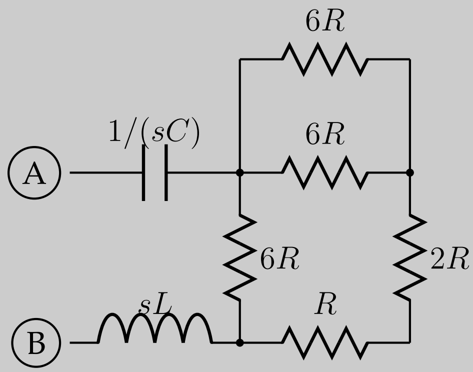

(b) Circuit for part (b)

Figure 1: Circuits for Question 1

1. **Circuit Equivalence**

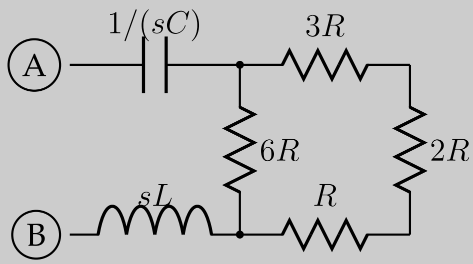
   (a) (4 points) Assuming zero initial conditions, find the impedance equivalent to the circuit in Figure 1(a) as seen from the terminals A and B. The answer should be given as the ratio of two polynomials.

   **Solution:**
   Assuming zero initial conditions, the circuit is equivalent to the $s$-domain circuit:



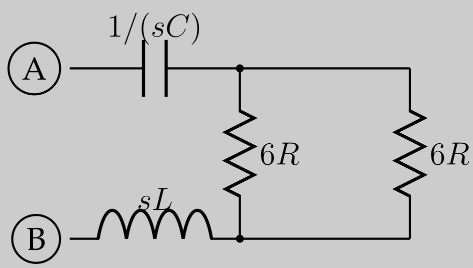

(+1 point)

Associating the two $6\ \Omega$ resistors in parallel we obtain



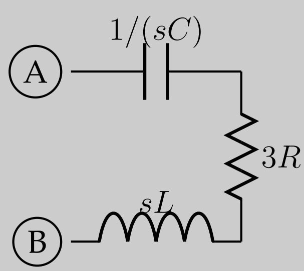

(+0.5 point)

Associating the $1$, $2$, and $3\ \Omega$ resistors in series we obtain



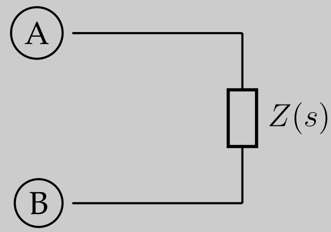

(+0.5 point)

Associating the two $6\ \Omega$ resistors in parallel we obtain




(+0.5 point)

Associating all impedances in series produces the equivalent impedance




where

$$Z(s) = 3R + sL + \frac{1}{sC} = \frac{1 + 3RCs + s^2L}{sC} \qquad \text{(+1 point)}$$

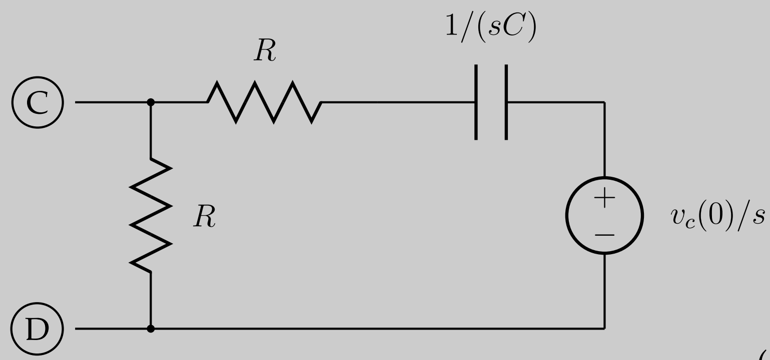
which is the following ratio of polynomials

$$Z(s) = \frac{1 + 3RCs + s^2LC}{sC} \qquad \text{(+0.5 point)}$$

(b) (4 points) Assuming that the initial condition on the capacitor is as indicated in the diagram, redraw the circuit shown in Figure 1(b) in the $s$-domain. Then use source transformations to find the $s$-domain Norton equivalent of this circuit as seen from the terminals C and D. The equivalent impedance and current source should be given as the ratio of two polynomials.

**Solution:**

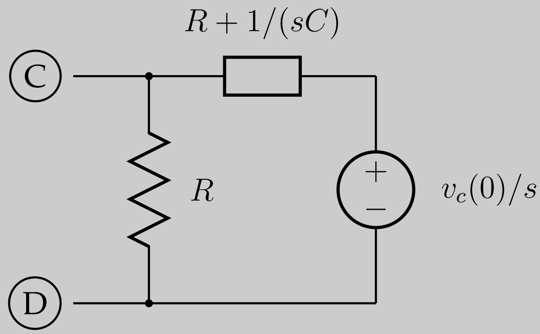
Assuming that the capacitor has initial condition $v_c(0)$ as indicated in the figure, the equivalent $s$-domain circuit is:




(+1 point)

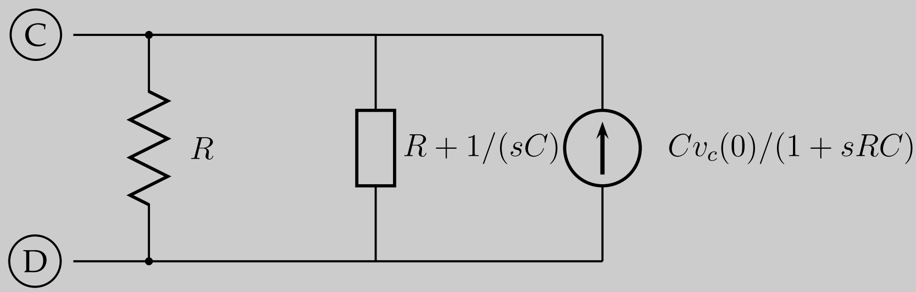
NOTE TO GRADERS: One can alternatively use a current source to represent the initial condition. Please grade the question accordingly.

Associating the resistor and capacitor in series we obtain:




(+0.5 point)

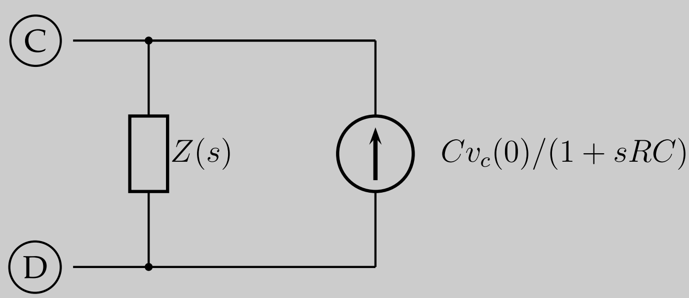
A source transformation produces:




where the current source value is calculated as

$$\frac{v_c(0)/s}{R + 1/(sC)} = \frac{Cv_c(0)}{1 + sRC}$$ (+ 1 point)

Finally associating the resistor with the impedance produces:




where

$$Z(s) = \frac{1}{\dfrac{1}{R} + \dfrac{sC}{1 + sRC}}$$ (+1 point)

which is the following ratio of polynomials

$$Z(s) = \frac{1}{\frac{1 + sRC + sRC}{R + sR^2C}} = \frac{R + sR^2C}{1 + 2sRC}$$

(+0.5 point)

Total for Question 1: 8

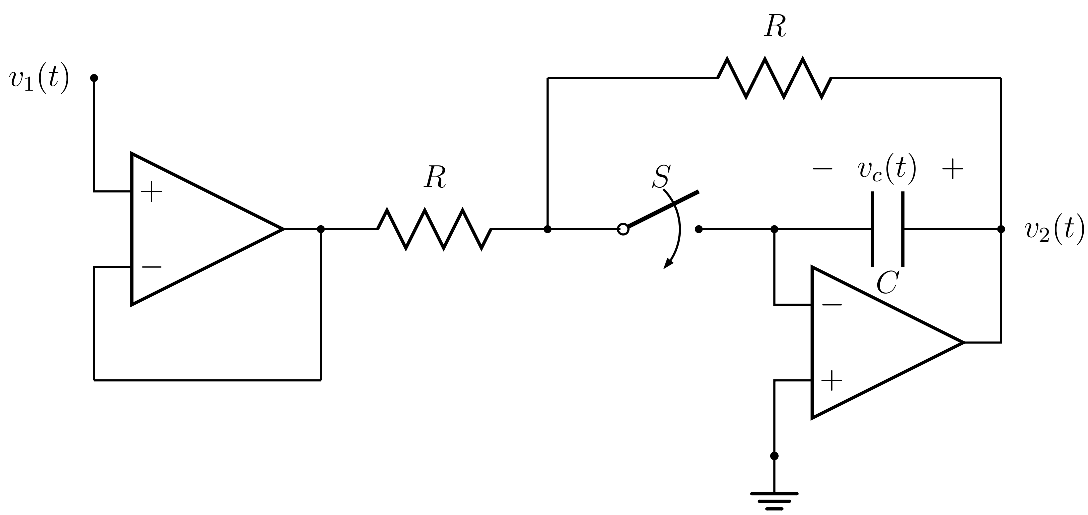



Figure 2: Circuit for Question 2

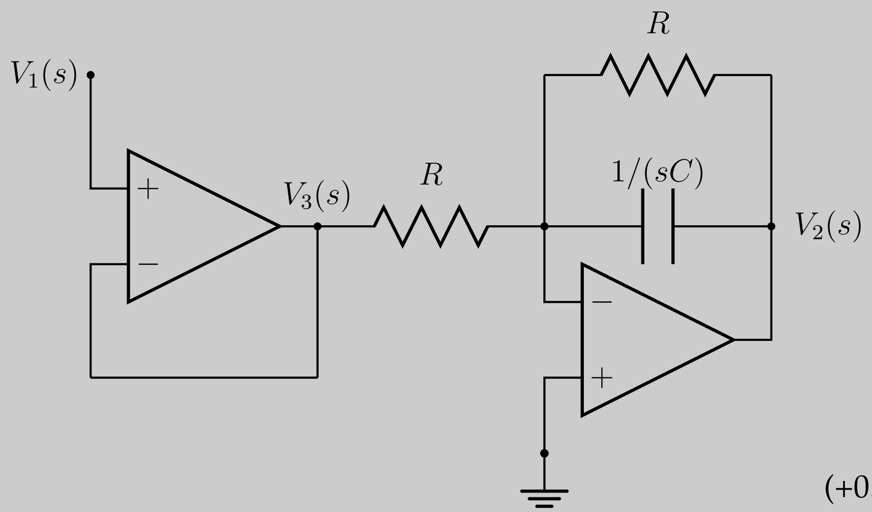
2. **Op-Amp Analysis in the $s$-Domain**

Consider the circuit in Figure 2.

(a) (3 points) Assume that the initial voltage on the capacitor is $v_c(0) = 0$ and that the switch $S$ is in the closed position. Convert the circuit into the $s$-domain and show that

$$V_2(s) = \frac{-\omega_c}{s + \omega_c} V_1(s), \quad \omega_c = \frac{1}{RC}.$$

**Solution:**
Assuming zero initial conditions, the circuit is equivalent to the $s$-domain circuit:




(+0.5 point)

From $V_1(s)$ to $V_3(s)$ we have a buffer, hence

$$V_3(s) = V_1(s) \qquad \text{(+1 point)}$$

From $V_3(s)$ to $V_2(s)$ we have a standard inverter op-Amp, hence

$$V_2(s) = -\frac{Z_1(s)}{R} V_3(s), \qquad Z_1(s) = \frac{1}{\frac{1}{R} + sC} = \frac{R}{1 + sRC} \qquad \text{(+1 point)}$$

Consequently

$$V_2(s) = -\frac{1/(RC)}{s + 1/(RC)} V_1(s) = \frac{-\omega_c}{s + \omega_c} V_1(s) \qquad \text{(+0.5 point)}$$

(b) (3 points) Assume that the initial voltage on the capacitor is $v_c(0) = 0$ and that $v_1(t)$ is a constant voltage source with voltage $V_1$ Volts. Show that

$$v_2(t) = \left(e^{-\omega_c t} - 1\right) V_1 u(t).$$

when the switch $S$ is in the closed position. *Hint: Use your answer from part (a).*

**Solution:**
Because $v_1(t)$ is a constant voltage source with voltage $V_1$ Volts we have that

$$V_1(s) = V_1/s$$

and using the answer from part (a)

$$V_2(s) = \frac{-\omega_c}{s + \omega_c} \frac{V_1}{s} \qquad \text{(+1 point)}$$

We look for a partial fraction expansion

$$\frac{\alpha}{s+\omega_c} + \frac{\beta}{s} = \frac{-\omega_c V_1}{s(s+\omega_c)}$$

where

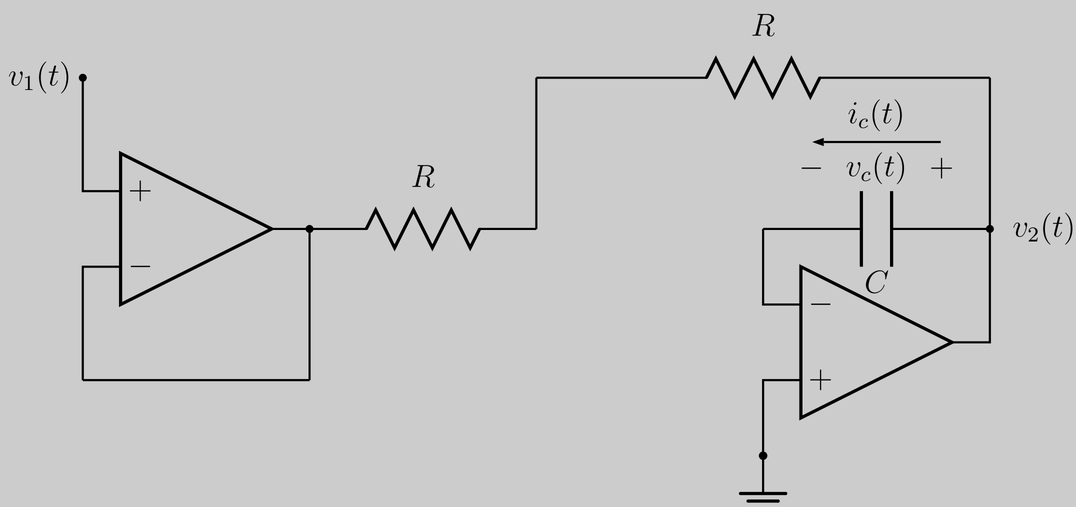
$$\alpha = \lim_{s\to-\omega_c} \frac{-\omega_c V_1}{s} = V_1, \qquad \beta = \lim_{s\to 0} \frac{-\omega_c V_1}{s+\omega_c} = -V_1 \qquad \text{(+1 point)}$$

which produce the inverse Laplace transform

$$v_2(t) = (\alpha e^{-\omega_c t} + \beta)u(t) = (e^{-\omega_c t} - 1)V_1 u(t). \qquad \text{(+1 point)}$$

(c) (3 points) Assume that $v_1(t)$ is a constant voltage source with voltage $V_1$ Volts and that the switch has been on the closed position for a long time. At $t = 0$ the switch is open. Compute $v_2(t)$ for $t \geq 0$. *Hint: Use your answer from part (b).*

**Solution:**
With the switch open, the circuit becomes




From part (a), after a long time

$$v_2^{(a)}(\infty) = \lim_{t\to\infty}(e^{-\omega_c t} - 1)V_1 u(t) = -V_1 \qquad \text{(+1 point)}$$

Because on both circuits $v_- = v_+ = 0$ on the second op-Amp, we have

$$v_c(t) = v_2(t) - v_- = v_2(t), \qquad v_c(0) = v_2^{(a)}(\infty) = -V_1. \qquad \text{(+1 point)}$$

Furthermore, since $i_c(t) = i_N = 0$ on the op-Amps, we have

$$v_c(t) = v_c(0) = v_2^{(a)}(\infty) = -V_1, \quad \text{for all } t \geq 0. \qquad \text{(+1 point)}$$

NOTE TO GRADERS: This question may be tricky to answer correctly. Exercise your best judgement to interpret the student answer and take their side if correct but not 100% precise.

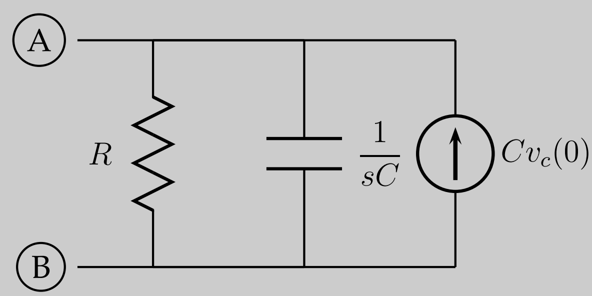
(d) (1 point) Justify why this circuit is called *sample-and-hold*?

**Solution:**
If $v_1(t)$ varies slowly, with the switch closed, $v_2(t) \to -v_1(t)$, therefore sampling the input. (+0.5 point)

With the switch open at $t = \tau$, $v_2(t) = -v_1(\tau)$, for all $t \geq \tau$, therefore holding the value of the input at the output. (+0.5 point)

(e) (3 points (bonus)) Repeat part (b), assuming this time that the initial voltage on the capacitor is $v_c(0) = V_0 \neq 0$.

**Solution:**
Start by finding the Thevenin equivalent of the circuit:



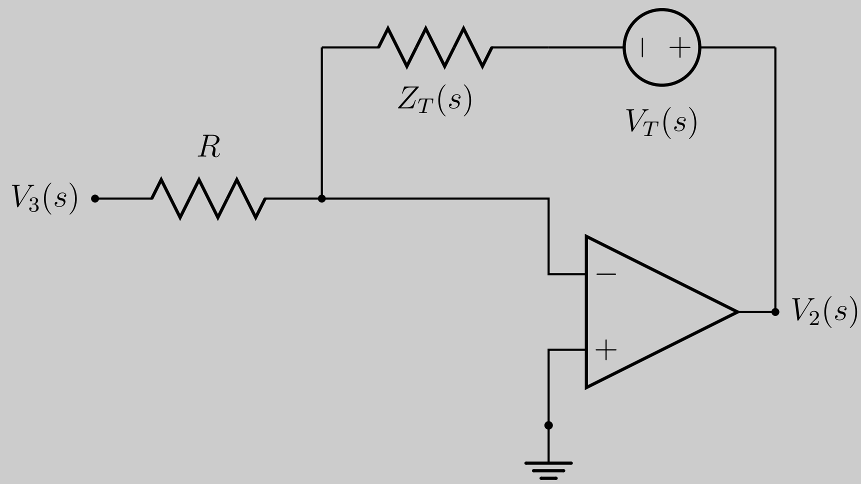

The Thevenin voltage is the open circuit voltage

$$V_T(s) = V_{AB}(s) = \frac{1}{\dfrac{1}{R} + sC} C v_c(0) = \frac{1}{s + \omega_c} v_c(0)$$

and the impedance is $R$ in paralell with $1/(sC)$

$$Z_T(s) = \frac{1}{\frac{1}{R} + sC} = \frac{R}{1 + sRC}$$ (+1 point)

Then we can analyze the inverter:




for which

$$V_2(s) = V_T(s) - Z_T(s)I(s), \qquad I(s) = \frac{V_3(s)}{R}$$

which leads to

$$V_2(s) = \frac{1}{s+\omega_c}v_c(0) - \frac{R}{1+sRC}\frac{V_3(s)}{R} = \frac{v_c(0)}{s+\omega_c} - \frac{\omega_c}{s+\omega_c}V_3(s) \qquad \text{(+1 point)}$$

(+1 point)

For $V_3(s) = V_1(s) = V_1/s$ we have that the second part of the response is the same we have already calculated whereas the first part yields

$$\mathcal{L}\left\{\frac{v_c(0)}{s+\omega_c}\right\} = v_c(0)e^{-\omega_c t}u(t)$$

for a total response of

$$v_2(t) = \left[(v_c(0)+V_1)e^{-\omega_c t} - V_1\right]u(t). \qquad \text{(+1 point)}$$

Total for Question 2: 10

Total for Question 2: 3 (bonus)

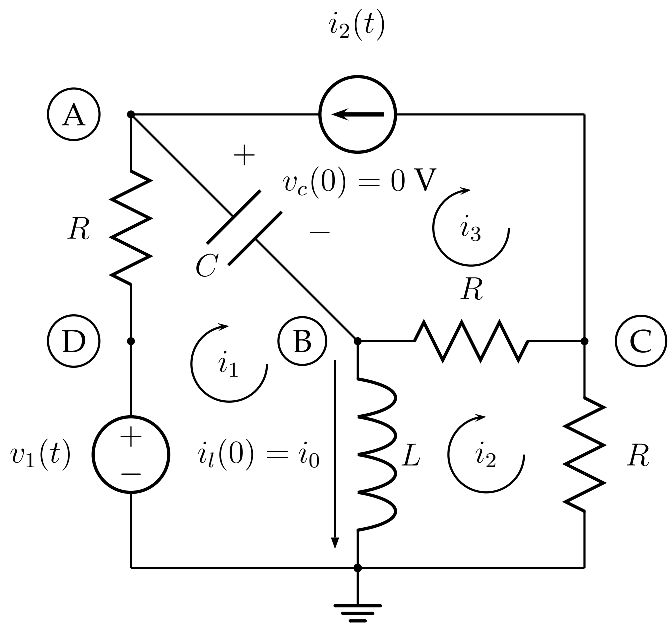



Figure 3: Circuit for Question 3

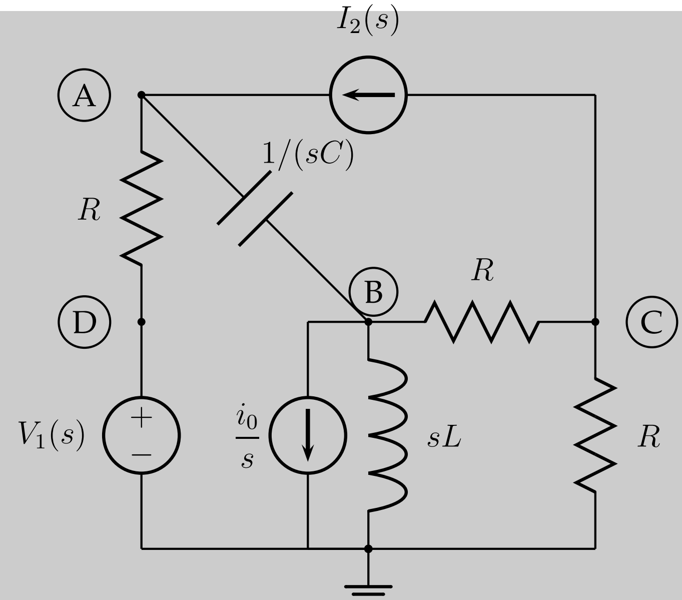
3. **Nodal and Mesh Analysis**

   In your answer to this question, clearly indicate the voltages, currents and the equations that need to be solved in each part.

   (a) (6 points) Formulate node-voltage equations in the $s$-domain for the circuit in Figure 3. Use the reference node and other labels as shown in the figure. Use the initial conditions indicated in the figure! Transform initial conditions on the capacitor and on the inductor into current sources.

   **Solution:**

   Note that the initial condition of the capacitor is zero and convert the circuit to the $s$-domain using a current source to represent the initial condition of the inductor to obtain:




(+1 point)

If the initial condition is represented correctly, add (+0.5 point)

Using method #2 to account for the voltage source

$$V_D(s) = V_1(s) \qquad \text{(+1 point)}$$

and not to write KCL @ node D, we obtain the node-voltage equations by inspection:

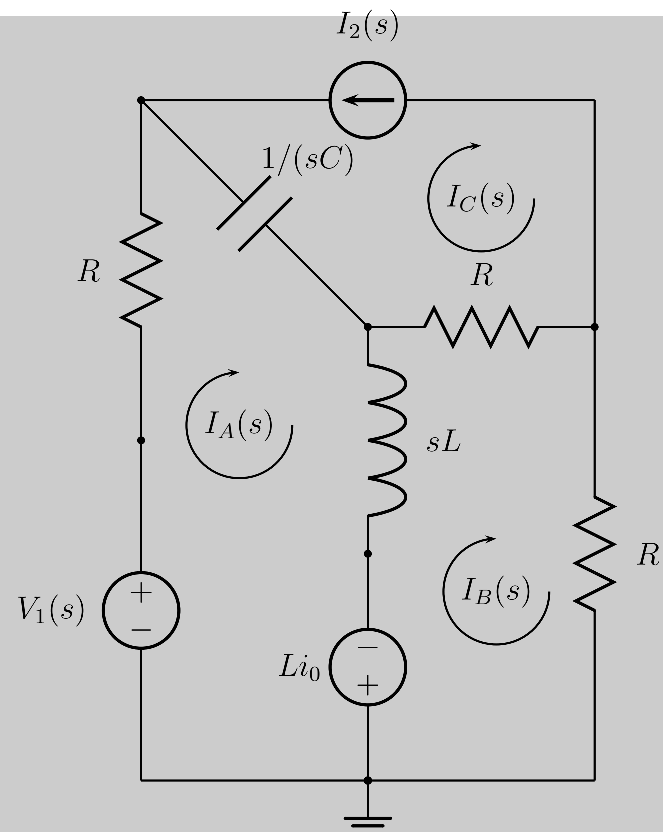
$$\begin{bmatrix} \frac{1}{R} + sC & -sC & 0 & -\frac{1}{R} \\ -sC & \frac{1}{R} + \frac{1}{sL} & -\frac{1}{R} & 0 \\ 0 & -\frac{1}{R} & \frac{2}{R} & 0 \end{bmatrix} \begin{pmatrix} V_A(s) \\ V_B(s) \\ V_C(s) \\ V_D(s) \end{pmatrix} = \begin{pmatrix} I_2(s) \\ -\frac{i_0}{s} \\ -I_2(s) \end{pmatrix} \qquad \text{(+3 points)}$$

The above 4 equations should be solved for the node-voltages (unknowns) $V_A(s)$, $V_B(s)$, $V_C(s)$ and $V_D(s)$. (+0.5 point)

(b) (6 points) Formulate mesh-current equations in the s-domain for the circuit in Figure 3. Use the mesh currents shown in the figure. Use the initial conditions indicated in the figure! Transform initial conditions on the capacitor and on the inductor into voltage sources.

**Solution:**

Note that the initial condition of the capacitor is zero and convert the circuit to the $s$-domain using a voltage source to represent the initial condition of the inductor to obtain:




(+1 point)

If the initial condition is represented correctly, add (+0.5 point)

NOTE TO GRADERS: The question has a potential notational conflict between the mesh current $i_2$ and the current source $I_2$ when converted to the $s$-domain. I instructed the students to use $i_A$, $i_B$, $i_C$ instead or not captalize the mesh currents.

Using method #2 to account for the current source

$$I_C(s) = -I_2(s) \qquad \text{(+1 point)}$$

and not to write KVL @ mesh C, we obtain the mesh-current equations by inspection:

$$\begin{bmatrix} R + \frac{1}{sC} + sL & -sL & -\frac{1}{sC} \\ -sL & 2R + sL & -R - \frac{1}{sC} \end{bmatrix} \begin{pmatrix} I_A(s) \\ I_B(s) \\ I_C(s) \end{pmatrix} = \begin{pmatrix} V_1(s) + Li_0 \\ -Li_0 \end{pmatrix} \qquad \text{(+3 points)}$$

The above 3 equations should be solved for the mesh-currents (unknowns) $I_A(s)$, $I_B(s)$ and $I_C(s)$. (+0.5 point)

Total for Question 3: 12

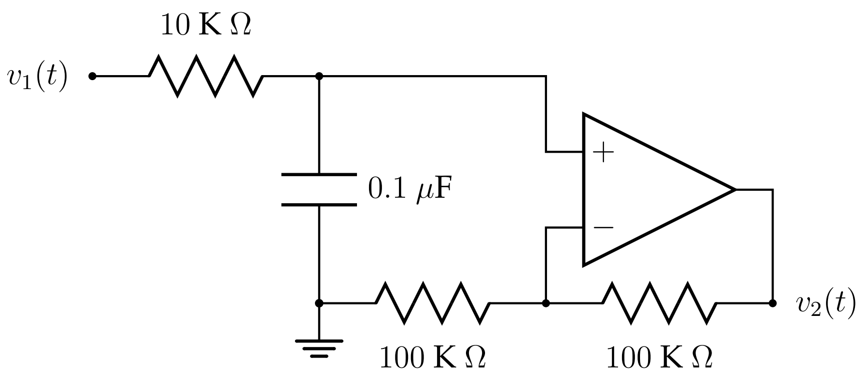


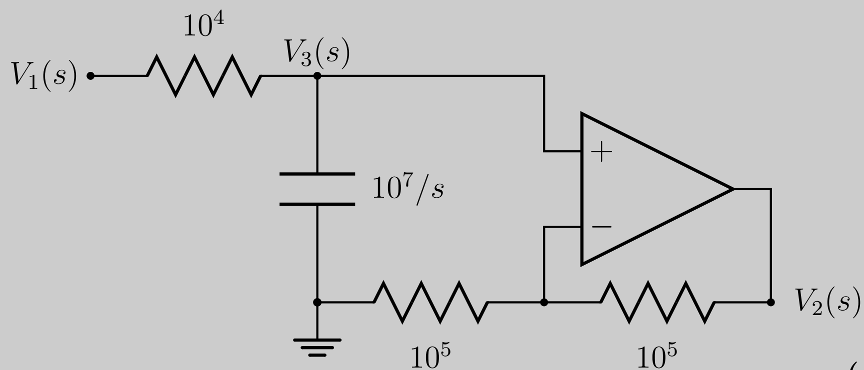

Figure 4: Circuit for Question 4

4. **Frequency Response**

(a) (3 points) Assuming zero initial conditions, transform the circuit in Figure 4 into the $s$-domain and compute the transfer function from $V_1(s)$ to $V_2(s)$.

**Solution:**

Assuming zero initial conditions and transforming the circuit to the $s$-domain we obtain:




(+1 point)

From $V_1(s)$ to $V_3(s)$ the circuit is a voltage divider

$$V_3(s) = \frac{10^7/s}{10^4 + 10^7/s} V_1(s) = \frac{10^3}{s + 10^3} V_1(s).$$

From $V_3(s)$ to $V_2(s)$ the circuit is a standard non-inverting op-Amp for which

$$V_2(s) = \frac{10^5 + 10^5}{10^5} V_3(s) = 2V_3(s).$$

Hence the transfer function is

$$T(s) = \frac{V_2(s)}{V_1(s)} = \frac{2 \times 10^3}{s + 10^3}.$$

(b) (3 points) Use what you know about frequency response to compute the steady-state response of the above circuit to the input $v_1(t) = 2\cos(10t)$.

**Solution:**
Using the frequency response method, we know that

$$v_2^{SS}(t) = 2|T(j10)|\cos(10t + \angle T(j10)) \qquad \text{(+1 point)}$$

Evaluating

$$T(j10) = \frac{2 \times 10^3}{j10 + 10^3} = \frac{2 \times 10^2}{j + 10^2} = \frac{200}{\sqrt{10001}} e^{-j\tan^{-1}(1/100)} \approx 2e^{-j/100} \qquad \text{(+1 point)}$$

Therefore

$$v_2^{SS}(t) = \frac{400}{\sqrt{10001}} \cos(10t - \tan^{-1}(1/100)) \approx 4\cos(10t - 1/100) \qquad \text{(+1 point)}$$

(c) (3 points) Sketch the plot of the magnitude of the frequency response of the circuit and classify the circuit as low-pass, high-pass or band-pass filter. Compute the cutoff-frequency and the gain of the filter.
*Hint: The gain (G) is the maximum value of the magnitude of the frequency response and the cutoff frequency is the frequency for which the magnitude of the frequency response is equal to $G/\sqrt{2}$.*

**Solution:**
At $t \to 0$

$$|T(j0)| = \frac{20}{10} = 2.$$

At $t \to \infty$

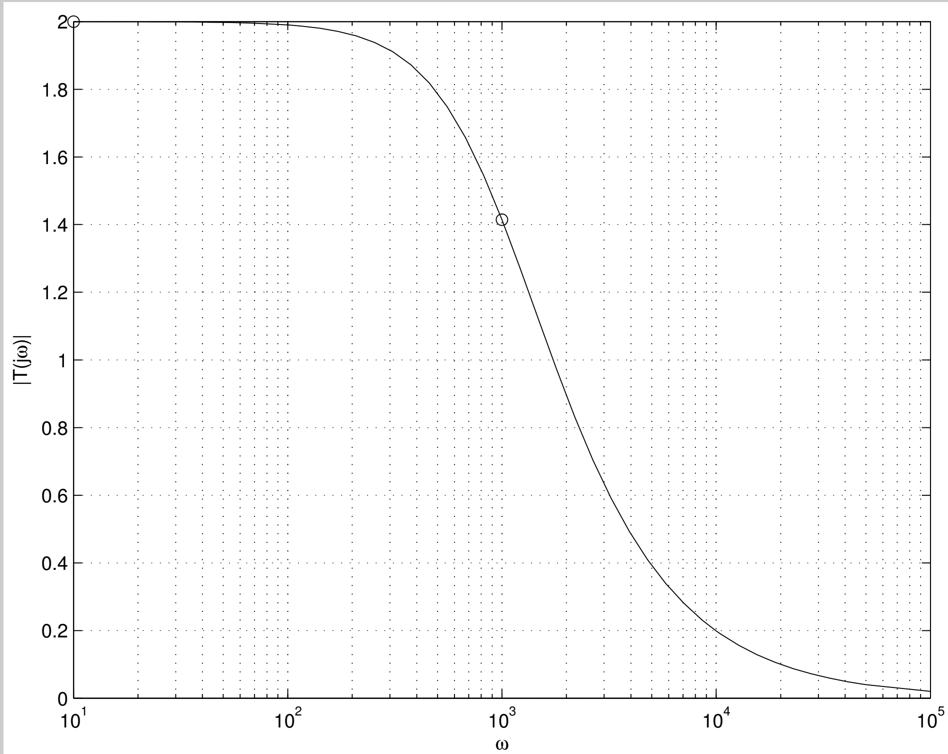
$$|T(j\infty)| \to 0$$

At $t \to 1000$

$$|T(j1000)| = \frac{2 \times 10^3}{\sqrt{2 \times 10^6}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

(+1 point)

A sketch of the magnitude of the frequency response should look like:




(+1 point)

The peak is at $j\omega = 0$ hence $G = 2$. Therefore the cutoff frequency is $\omega_c = 1000$ since $|T(j1000)| = G/\sqrt{2} = \sqrt{2}$. (+1 point)

Total for Question 4: 9

5. **Circuit Design**

In control systems, a proportional-integral controller is a system that computes the error between a voltage $v(t)$ and a given reference voltage $\bar{v}(t)$, i.e.

$$e(t) = v(t) - \bar{v}(t) \tag{1}$$

and then adds two signals that are proportional to the error

$$u_P(t) = K_P\, e(t), \tag{2}$$

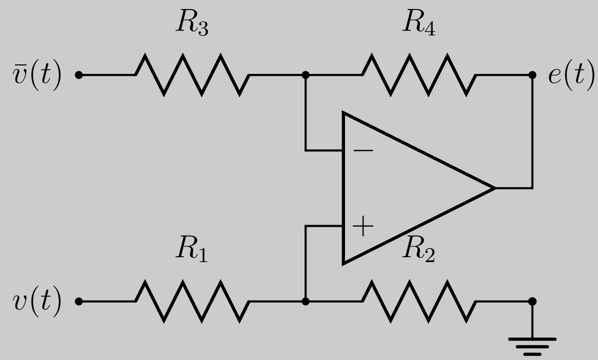
and its integral

$$u_I(t) = K_I \int_0^t e(\tau)\, d\tau. \tag{3}$$

where $K_P \geq 0$ and $K_I \geq 0$ are given control gains.

(a) (3 points) Design an op-amp circuit that performs operation (1). Assume that the voltages $v(t)$ and $\bar{v}(t)$ are available. Clearly indicate how to select the values of the components you used in your circuit.

**Solution:**
We can use a subtractor




(+1 point)

for which

$$e(t) = \frac{R_2}{R_1 + R_2}\frac{R_3 + R_4}{R_3}v(t) - \frac{R_4}{R_3}\bar{v}(t) \qquad \text{(+1 point)}$$

For $e(t) = v(t) - \bar{v}(t)$ we choose

$$R_3 = R_4, \qquad R_1 = R_2 \qquad \text{(+1 point)}$$

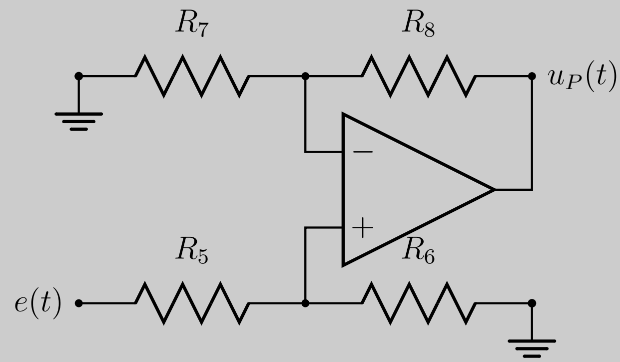
and

$$\frac{R_4}{R_3} = 1, \qquad \frac{R_2}{R_1 + R_2}\frac{R_3 + R_4}{R_3} = 1.$$

NOTE TO GRADERS: Some students might prefer to give concrete numerical values for the components.
NOTE TO GRADERS: Any other equivalent design is also correct, even if it has more OpAmps.

(b) (3 points) Design an op-amp circuit that performs operation (2). Assume that the voltage $e(t)$ is available. Clearly indicate how to select the values of the components you used in your circuit. What is the possible range of values for $K_P$?

**Solution:**
The same circuit as before with one of voltages equal to zero, which is a non-inverting amplifier preceeded by a voltage divider works here




(+1 point)

for which

$$u_P(t) = \frac{R_6}{R_5 + R_6} \frac{R_7 + R_8}{R_7} e(t) \qquad \text{(+1 point)}$$

The resistors $R_5$, $R_6$, $R_7$ and $R_8$ are chosen so that

$$\frac{R_6}{R_5 + R_6} \frac{R_7 + R_8}{R_7} = K_P \geq 0 \qquad \text{(+1 point)}$$

Note that the voltage divider is needed if we want to set $0 \leq K_P \leq 1$.

NOTE TO GRADERS: Please consider a non-inverting setup also fine here even though $K_P \geq 1$. Discount -1 point if $u_P(t) = -K_P\, e(t)$.

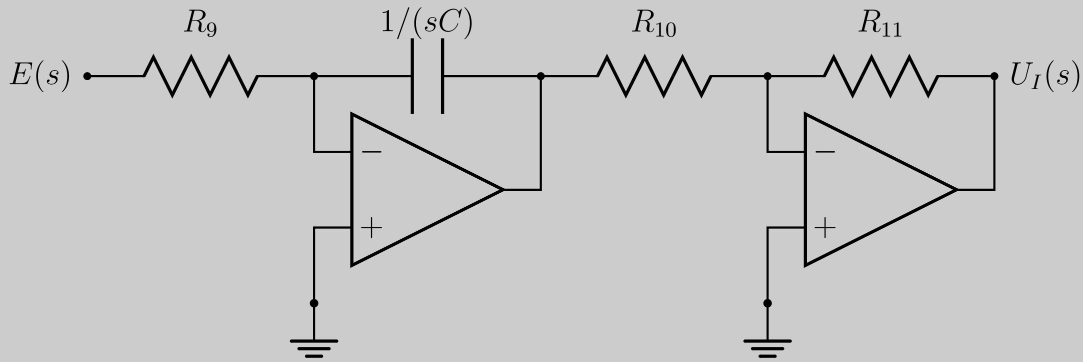
(c) (3 points) Design an op-amp circuit that performs operation (3). Assume that the voltage $e(t)$ is available. Clearly indicate how to select the values of the components you used in your circuit. What is the possible range of values for $K_I$? *Hint: First compute the transfer function between $E(s)$ and $U_I(s)$.*

**Solution:**

Taking Laplace transforms

$$U_I(s) = \mathcal{L}\{u_I(t)\} = K_I \mathcal{L}\left\{\int_0^t e(\tau)\, d\tau\right\} = \frac{K_I}{s} E(s) \qquad \text{(+1 point)}$$

Perhaps the easiest way to implement this circuit is with a inverter Op-Amp, followed by another inverter. In the $s$-domain we have




for which

$$U_I(s) = \frac{-R_{11}}{R_{10}} \times \frac{-1/(sC)}{R_9} E(s) = \frac{R_{11}}{sR_9R_{10}C} E(s) \qquad \text{(+1 point)}$$

The resistors $R_9$, $R_{10}$, $R_{11}$ and the capacitor $C$ are chosen so that

$$\frac{R_{11}}{R_9R_{10}C} = K_I \geq 0 \qquad \text{(+1 point)}$$

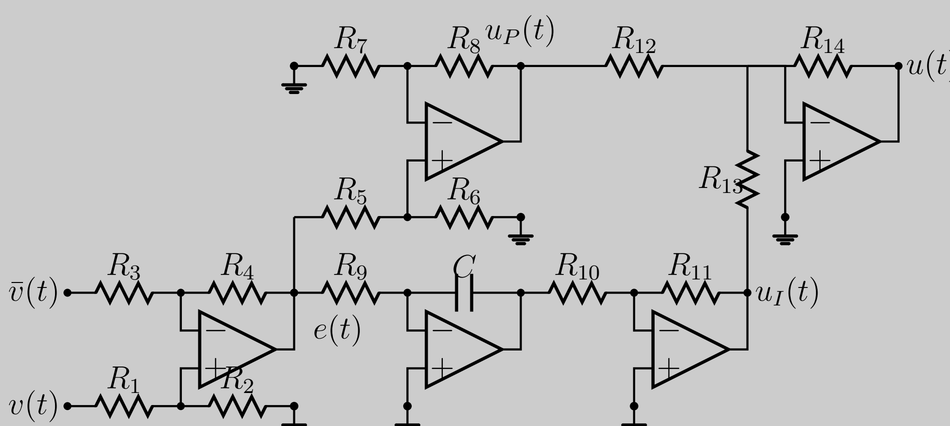
Note that the second inverter in order to have $K_P \geq 0$.

(d) (3 points) Combine the three circuits you designed in parts (a)-(c) to produce

$$u(t) = -K_P[v(t) - \bar{v}(t)] - K_I \int_0^t [v(\tau) - \bar{v}(\tau)]\, d\tau.$$

Pay attention to the negative signs!

**Solution:**
We can combine all circuits into a single circuit followed by an inverter summer to add the terms. The inverter sum will take care of the signal inversion. Here is what it would like:




(+1 point)

This combination produces

$$u(t) = -\frac{R_{14}}{R_{12}} u_P(t) - \frac{R_{14}}{R_{13}} u_I(t) \qquad \text{(+1 point)}$$

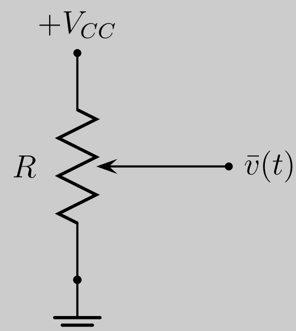
For $u(t) = -u_P(t) - u_I(t)$ the resistors $R_{12}$, $R_{13}$, $R_{14}$ are chosen so that

$$R_{12} = R_{13} = R_{14}. \qquad \text{(+1 point)}$$

(e) (3 points) Assume that the reference voltage $\bar{v}(t)$ is produced with the help of a potentiometer used as a voltage-divider. Modify the circuit you designed in part (a) to accommodate for this fact. If you circuit needs no modifications, explain why.

**Solution:**

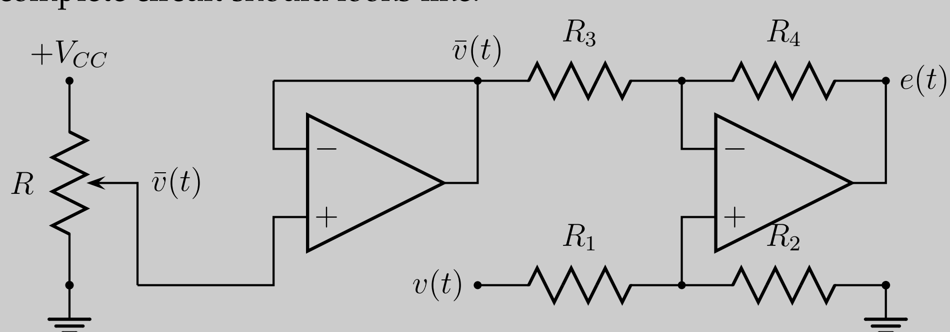
If the voltage $\bar{v}(t)$ is produced with a voltage divider we have




(+1 point)

Because the input $\bar{v}(t)$ has low impedance, we need a buffer to decouple the voltage divider from the subtractor. (+1 point)

The complete circuit should looks like:

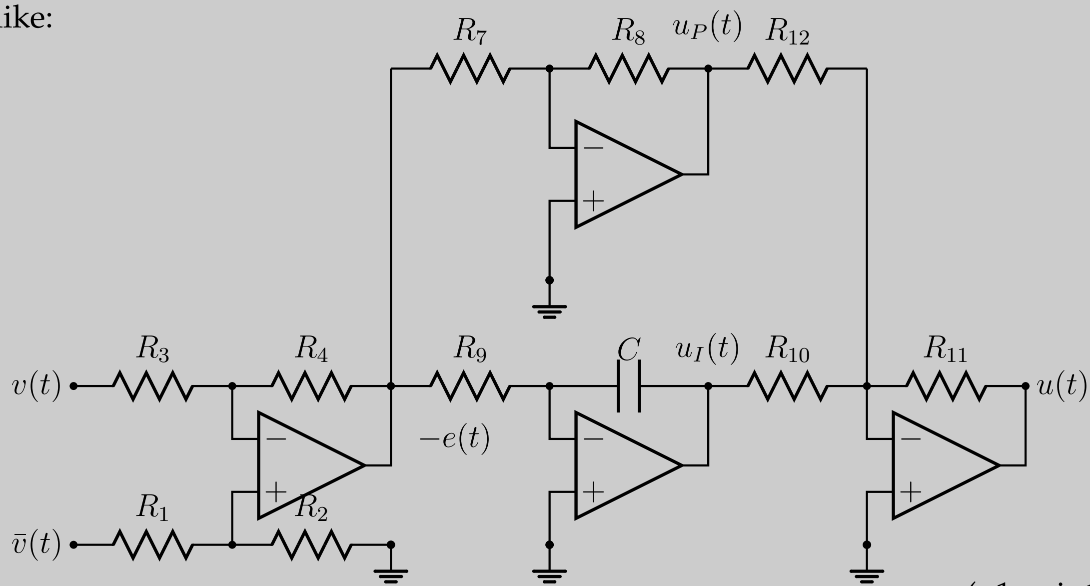



(+1 point)

(f) (1 point (bonus)) Redesign the circuit you obtained in part (d) to use at most four Op-Amps. If your design already uses at most four op-Amps you automatically get the bonus point.

**Solution:**

We can invert $e(t)$ and generate $u_p(t)$ and $u_I(t)$ with an inverter block and combine them all with a single inverter Op-Amp. The resulting circuit should look like:




(+1 point)

The set of components is now:

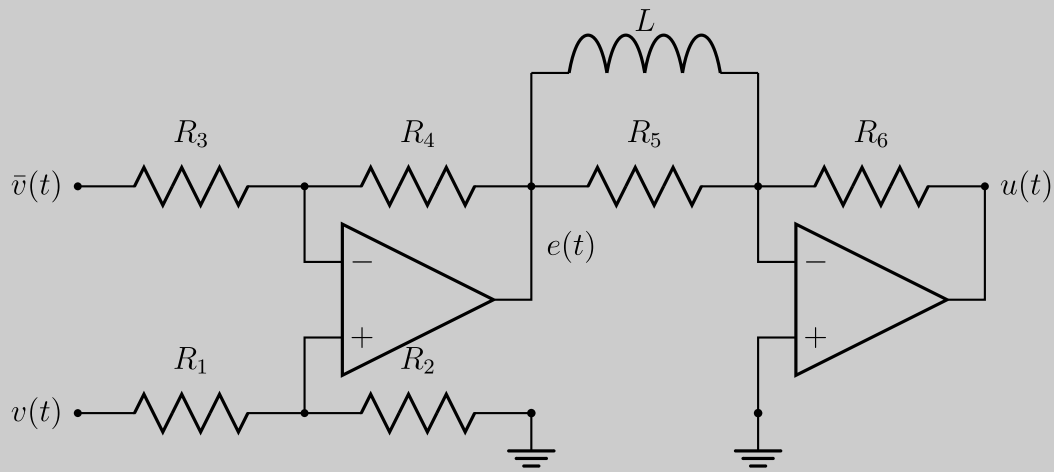
$$R_1 = R_2 = R_3 = R_4, \qquad R_{10} = R_{11} = R_{12},$$

$$\frac{R_8}{R_7} = K_P, \qquad \frac{1}{R_9 C} = K_I.$$

(g) (2 points (bonus)) Redesign the circuit you obtained in part (d) to use at most three Op-Amps. If your design already uses at most three op-Amps you automatically get the bonus points.

**Solution:**

Here is a solution that uses only two Op-Amps and an inductor:




The set of components is now:

$$R_1 = R_2 = R_3 = R_4, \qquad \frac{R_6}{R_5} = K_P, \qquad \frac{R_6}{L} = K_I.$$

This setup is hard to tune because of the nonlinear relationship of the resistors with the gains.

(+2 points)

Total for Question 5: 15

Total for Question 5: 3 (bonus)