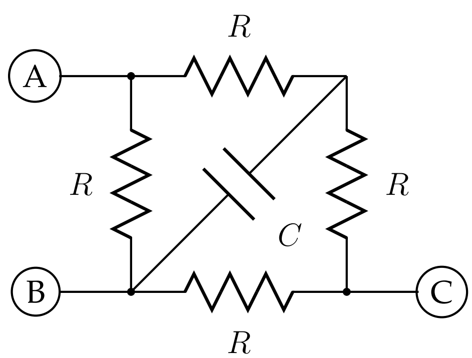
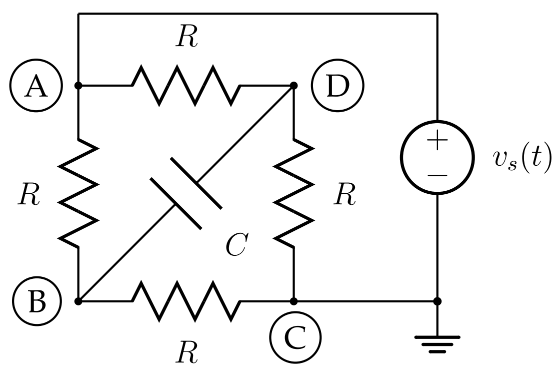
MAE140 – Linear Circuits – Winter – 2012
Final, March 20th

Instructions

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.
2. You have 170 minutes.
3. Do not forget to write your name and student number.
4. This exam has 3 questions, for a total of 38 points and 4 bonus points.



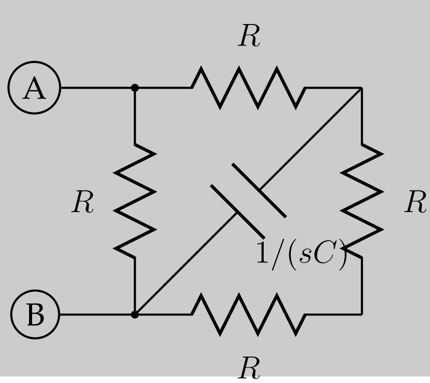
(a) Question 1 (a)



(b) Question 1 (b-e)

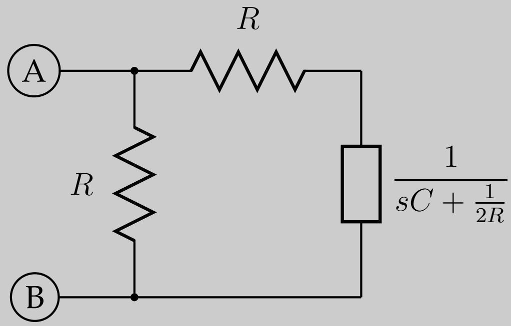
1. **Circuit Equivalence and Nodal Analysis**
   (a) (3 points) Assuming zero initial conditions, find the impedance equivalent to the circuit in Figure 1(a) as seen from terminals A and B. The answer should be given as the ratio of two polynomials.

Solution:
Assuming zero initial conditions, the circuit is equivalent to the s-domain circuit:



(+0.5 point)

Associating the two $R\,\Omega$ resistors in series with the capacitor $1/(sC)$ in parallel we obtain



(+1 point)

Associating the impedance with the $R\,\Omega$ resistor in series with the resistor R in parallel we obtain an equivalent impedance

$$Z(s) = \frac{1}{\frac{1}{R} + \frac{1}{R + \frac{1}{sC + \frac{1}{2R}}}}$$

(+1 point)

which is the following ratio of polynomials

$$\begin{aligned}
Z(s) &= \frac{1}{\frac{1}{R} + \frac{1}{R + \frac{2R}{1+s2RC}}} \\
&= \frac{1}{\frac{1}{R} + \frac{1+s2RC}{3R+s2R^2C}} \\
&= \frac{R(3R + s2R^2C)}{3R + s2R^2C + R + s2R^2C} \\
&= \frac{3R + s2R^2C}{4R + s4RC}
\end{aligned}$$

(+0.5 point)

(b) (1 point (bonus)) In the next parts you will compute the impedance as seen from terminals A and C. Before you do it, explain why this is a more difficult problem.

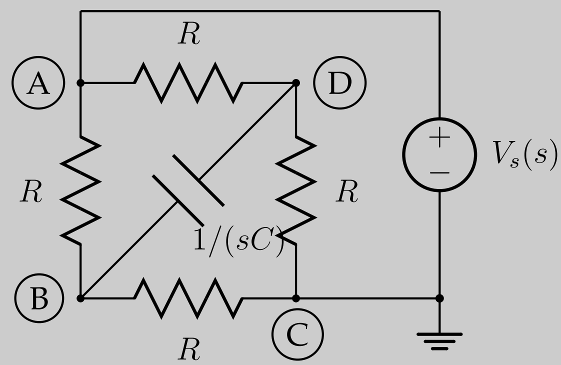
Solution: Because no elements are in series or parallel, hence no simplification is possible.

(+1 bonus point)

(c) (5 points) Connect a voltage source $v_s(t)$ as shown in Figure 1(b). Assuming zero initial conditions, redraw the circuit in the s-domain and formulate node-voltage equations in the s-domain. Use the reference node and other labels as shown in the figure. Clearly indicate the voltages and equations that need to be solved for.

Solution:

Converting the circuit to the s-domain we obtain



(+1 point)

We use Method # 2 to avoid writing KCL at node A. Instead we write the voltage source relationship

$$V_A(s) = V_s(s). \qquad \text{(+1 point)}$$

By inspection we write KCL at nodes B and D

$$\begin{bmatrix} -1/R & 2/R + sC & -sC \\ -1/R & -sC & 2/R + sC \end{bmatrix} \begin{pmatrix} V_A(s) \\ V_B(s) \\ V_D(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{(+2 points)}$$

The above three equations should be solved for the three unknown node voltages $V_A(s)$, $V_B(s)$, and $V_D(s)$.
(+1 point)

(d) (2 points) Show that

$$V_B(s) = V_D(s) = \frac{V_s(s)}{2}.$$

Hint: you may want to use the fact that

$$\begin{bmatrix} 2/R + Cs & -Cs \\ -Cs & 2/R + Cs \end{bmatrix}^{-1} = \frac{R}{4(1 + RCs)} \begin{bmatrix} 2 + RCs & RCs \\ RCs & 2 + RCs \end{bmatrix}.$$

Solution:
Substituting $V_A(s) = V_s(s)$ into the remaining KCL equations we obtain

$$\begin{bmatrix} 2/R + sC & -sC \\ -sC & 2/R + sC \end{bmatrix} \begin{pmatrix} V_B(s) \\ V_D(s) \end{pmatrix} = \begin{pmatrix} 1/R \\ 1/R \end{pmatrix} V_s(s) \qquad \text{(+1 point)}$$

Using the hint

$$\begin{pmatrix} V_B(s) \\ V_D(s) \end{pmatrix} = \begin{bmatrix} 2/R + sC & -sC \\ -sC & 2/R + sC \end{bmatrix}^{-1} \begin{pmatrix} 1/R \\ 1/R \end{pmatrix} V_s(s)$$

$$= \frac{R}{4(1+RCs)} \begin{bmatrix} 2 + RCs & RCs \\ RCs & 2 + RCs \end{bmatrix} \begin{pmatrix} 1/R \\ 1/R \end{pmatrix} V_s(s)$$

$$= \frac{1}{4(1+RCs)} \begin{pmatrix} 2 + 2RCs \\ 2 + 2RCs \end{pmatrix} V_s(s) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} V_s(s). \qquad \text{(+1 point)}$$

(e) (2 points) Using the results of the analysis you performed in parts (b)-(d), show that the impedance as seen from terminals A and C is $Z(s) = R$.
Hint: use the result of part (d).

Solution:
We writing KCL node A to obtain $I_s(s)$, the current on the source $V_s(s)$, as

$$I_s(s) = \frac{1}{R}[V_A(s) - V_B(s)] + \frac{1}{R}[V_A(s) - V_D(s)] = \frac{1}{R}V_s(s) \qquad \text{(+1 point)}$$

From $I_s(s)$ we compute

$$Z(s) = V_s(s)/I_s(s) = R \qquad \text{(+1 point)}$$

(f) (1 point (bonus)) Explain why $Z(s)$ does not depend on the capacitor C?

Solution:
Because the $V_B(s) = V_D(s)$, the current on the capacitor is zero no matter what source is connected to the circuit.
(+1 point)

Total for Question 1: 12

Total for Question 1: 2 (bonus)

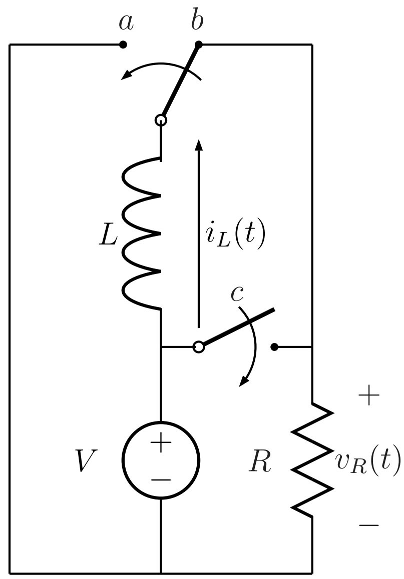


Figure 2: Circuit for Question 2

2. **Circuit Analysis in the s-Domain**

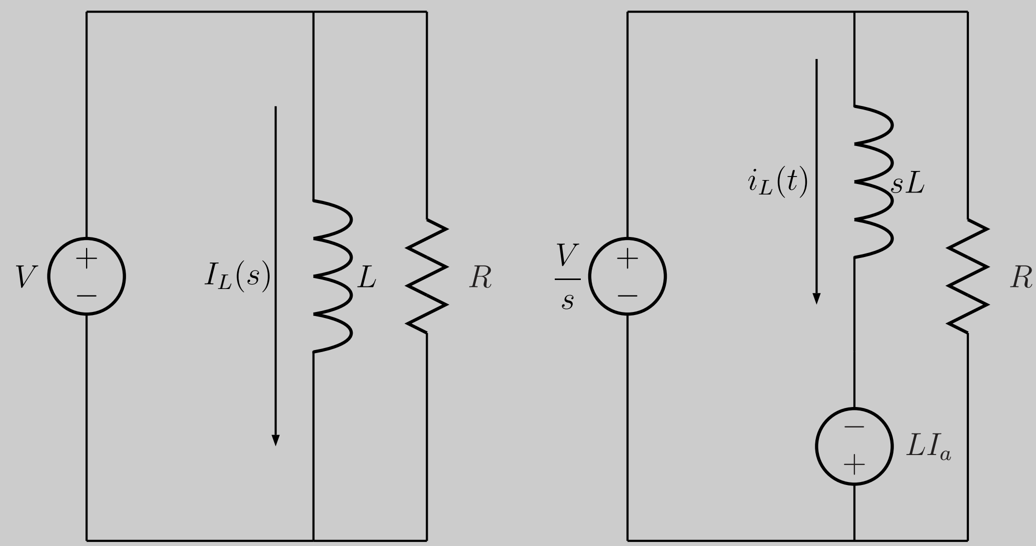
   The circuit in Figure 2 is a simplified *boost* circuit.

   (a) (3 points) At time $t = 0$ the switch ab is brought to position a and the switch c is closed. Assume that the initial current on the inductor at time $t = 0$ is $i_L(0) = I_a$. Show that the current on the inductor for $t \geq 0$ is

$$i_L(t) = \left(I_a + \frac{V}{L}\, t \right) u(t).$$

Solution:

In this configuration the circuit in the t-domain for $t \geq 0$ and its s-domain equivalent becomes



(+1 point)

from which

$$I_L(s) = \frac{1}{sL}\left(\frac{V}{s} + LI_a\right) = \frac{V}{s^2L} + \frac{I_a}{s} \qquad \text{(+1 point)}$$

Computing the Laplace inverse transform we obtain

$$i_L(t) = \left(I_a + \frac{V}{L}\,t\right)u(t) \qquad \text{(+1 point)}$$

Note also that

$$v_R(t) = Vu(t)$$

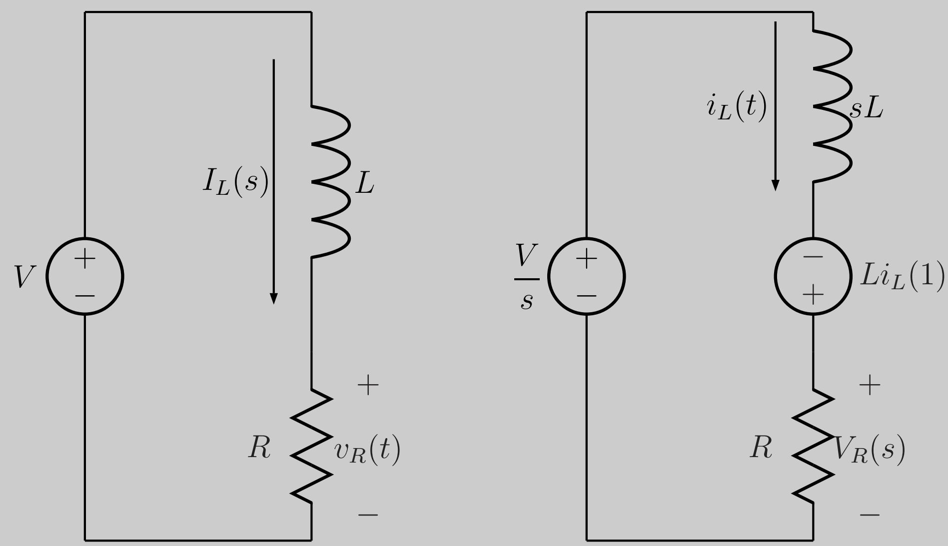
which is a fact we will use later.

(b) (3 points) At $t = 1$ s the switch is brought to position b and the switch c is open. Assume that $L = R$ and show that the voltage across the resistor R for $t \geq 1$ is equal to

$$v_R(t) = \left(V + RI_ae^{-(t-1)}\right)u(t-1)$$

Hint: do not forget to use $L = R$ and *the answer to part (a) to compute* $i_L(1)$.

Solution:
The circuit in the t-domain for $t \geq 1$ and its s-domain equivalent is simply



(+1 point)

The voltage at the resistor can be computed using the voltage-divider formula from which

$$V_R(s) = \frac{R}{sL+R}\left(\frac{V}{s} + Li_L(1)\right) = (R/L)\frac{V + Li_L(1)s}{s(s+R/L)} \qquad \text{(+0.5 point)}$$

With $L = R$

$$V_R(s) = \frac{V + Ri_L(1)s}{s(s+1)} \qquad \text{(+0.5 point)}$$

Expanding in partial fractions we obtain

$$V_R(s) = \frac{k_1}{s} + \frac{k_2}{s+1}$$

where

$$k_1 = \lim_{s\to 0} sV_R(s) = V, \qquad k_2 = \lim_{s\to -1} sV_R(s) = -[V - Ri_L(1)] \qquad \text{(+0.5 point)}$$

Substituting $i_L(1)$ from part (a)

$$i_L(1) = I_a + \frac{V}{L}$$

we have $k_2 = RI_a$. Computing the Laplace inverse transform we obtain

$$v_R(t) = \left(V + RI_a e^{-(t-1)}\right) u(t-1). \qquad \text{(+0.5 point)}$$

after recalling that the time has been shifted to $t \geq 1$.

(c) (2 points) Show that if

$$I_a = \frac{1}{1-e^{-1}} \frac{V}{R}$$

then at $t = 2$ s the current at the inductor and resistor are $i_R(2) = i_L(2) = I_a$. *Hint: use the answer to part (b).*

Solution:
From the circuit and the answer to part (b) we have

$$i_L(t) = i_R(t) = \frac{1}{R} v_R(t) = \left(\frac{V}{R} + I_a e^{-(t-1)} \right) u(t-1). \qquad \text{(+1 point)}$$

At $t = 2$

$$i_L(2) = i_R(2) = \frac{V}{R} + I_a e^{-1} = \left(1 + \frac{e^{-1}}{1-e^{-1}} \right) \frac{V}{R} = \frac{1}{1-e^{-1}} \frac{V}{R} = I_a. \qquad \text{(+1 point)}$$

(d) (2 points) Assume that I_a is as in part (c). Let the switch ab be brought to position a and the switch c be closed at time $t = 2$ s and remain there for one second. At $t = 3$ s the switch is brought to position b and the switch c is open. This operation is repeated in a periodic fashion. Sketch the voltage $v_R(t)$ on the resistor.

Solution:
With I_a as in part (c) the initial condition of the inductor at time $t = 2$ s is exactly the same as at time $t = 0$. Therefore, the response from $t = 2$ s to $t = 3$ s is the same as the response from $t = 0$ to $t = 1$ s. In these intervals

$$v_R(t) = V, \qquad 2\ell \le t < 2\ell + 1, \qquad \ell = 0, 1, \cdots \qquad \text{(+0.5 point)}$$

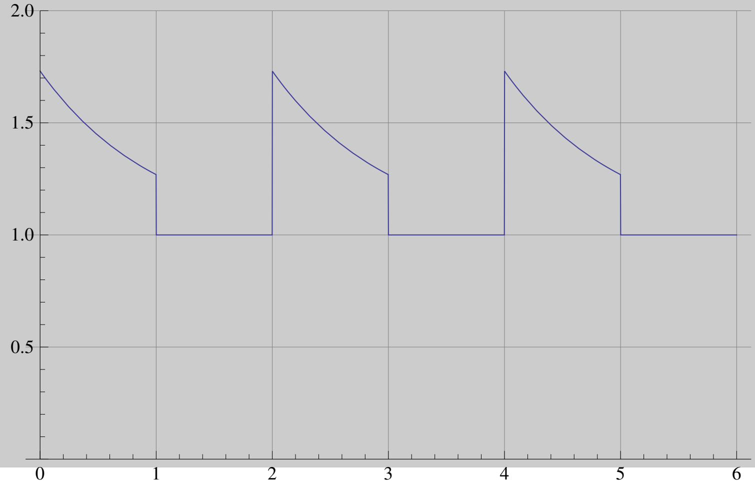
Following the same reasoning the response from $t = 3$ to $t = 4$ s is the same as the response from $t = 1$ to $t = 2$ s. That is

$$\begin{aligned} v_R(t) &= V + R I_a e^{-(t-2\ell-1)}, \\ &= V \left(1 + \frac{e^{-(t-2\ell-1)}}{1-e^{-1}} \right), \quad 2\ell + 1 \le t < 2\ell + 2, \quad \ell = 0, 1, \cdots \end{aligned} \qquad \text{(+0.5 point)}$$

At $t = 2\ell + 1$ we have

$$\begin{aligned} v_R(2\ell + 1) &= V \left(1 + \frac{1}{1-e^{-1}} \right) \approx V 1.73, \\ v_R(2\ell + 2^-) &= V \left(1 + \frac{e^{-1}}{1-e^{-1}} \right) \approx V 1.27. \end{aligned} \qquad \text{(+0.5 point)}$$

A plot of $v_R(t)$ for $V = 1$ is



(+0.5 point)

(e) (1 point (bonus)) Justify why this circuit is called a *boost* circuit? What is been boosted?

Solution:
It "boosts" the voltage on the resistor R to a level above the voltage source. On average, the voltage $v_R(t)$ is higher than V.

Total for Question 2: 10

Total for Question 2: 1 (bonus)

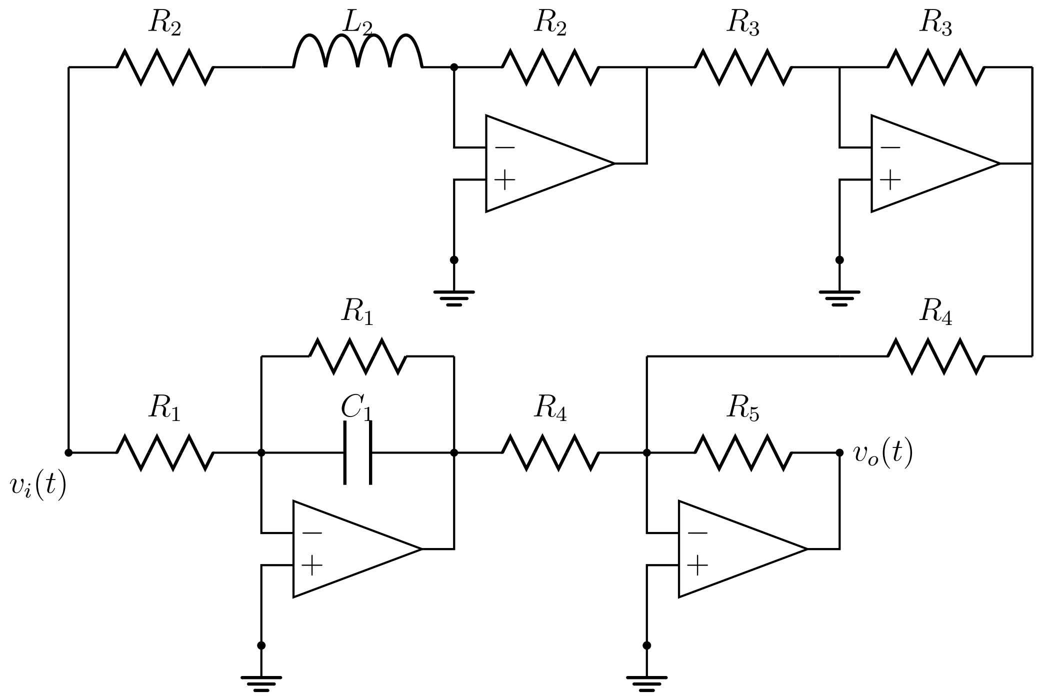


Figure 3: Circuit for Question 3

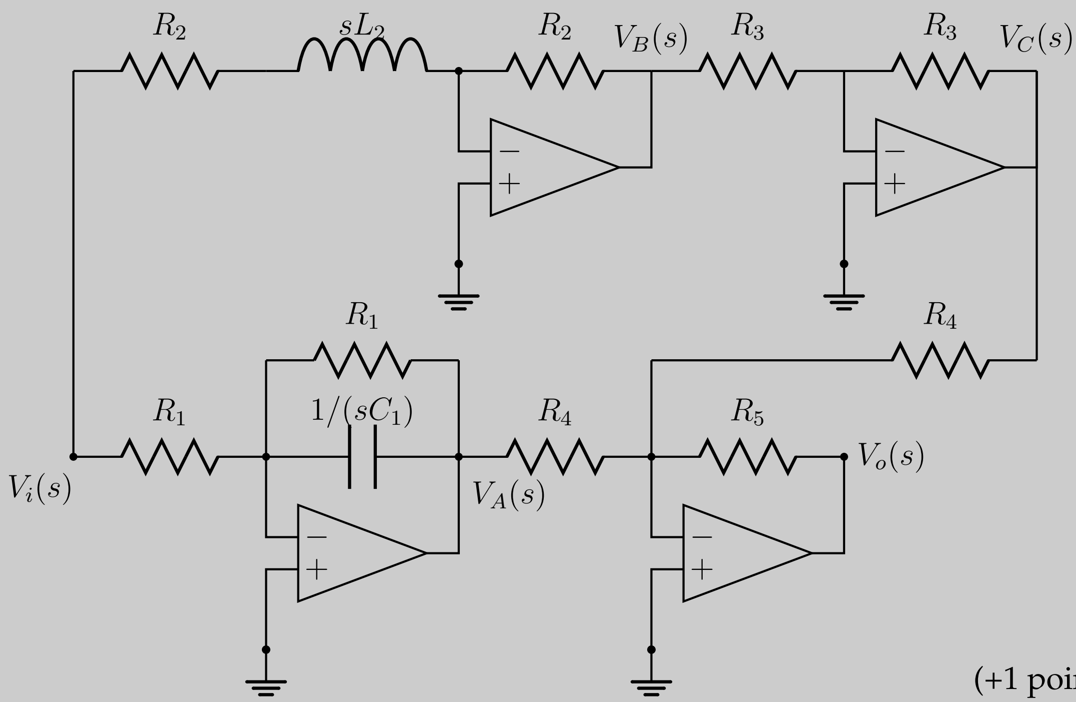
3. **Frequency Response and Circuit Design**

   (a) (5 points) Assuming zero initial conditions, transform the circuit in Figure 3 into the s-domain and show that the transfer function from $V_i(s)$ to $V_o(s)$ is of the form

$$\frac{V_o(s)}{V_i(s)} = \pm \frac{\alpha s}{(s+\omega_1)(s+\omega_2)}.$$

Solution:

Converting the circuit to the s-domain we obtain



(+1 point)

After recognizing that the circuit is made of simple inverter OpAmp modules we can write the following relationships

$$V_A(s) = -\frac{1}{R_1}\frac{1}{sC_1 + \frac{1}{R_1}}V_i(s) = -\frac{\omega_1}{s+\omega_1}V_i(s), \qquad \omega_1 = \frac{1}{R_1C_1}, \qquad \text{(+1 point)}$$

$$V_B(s) = -\frac{R_2}{sL_2 + R_2}V_i(s) = -\frac{\omega_2}{s+\omega_2}V_i(s), \qquad \omega_2 = \frac{R_2}{L_2}, \qquad \text{(+1 point)}$$

and

$$V_C(s) = -\frac{R_3}{R_3}V_B(s) = -V_B(s), \qquad \text{(+0.5 point)}$$

$$V_o(s) = -\frac{R_5}{R_4}[V_C(s) + V_A(s)] \qquad \text{(+0.5 point)}$$

Putting it all together

$$V_o(s) = -\frac{R_5}{R_4}\left[V_C(s) + V_A(s)\right] = \frac{R_5}{R_4}\left[V_B(s) - V_A(s)\right] \tag{1}$$

from where

$$\begin{aligned}
V_o(s) &= \frac{R_5}{R_4}\left[\frac{\omega_1}{s+\omega_1} - \frac{\omega_2}{s+\omega_2}\right]V_i(s) \\
&= \frac{R_5}{R_4}\left[\frac{\omega_1(s+\omega_2) - \omega_2(s+\omega_1)}{(s+\omega_1)(s+\omega_2)}\right]V_i(s) \\
&= \frac{R_5}{R_4}\left[\frac{(\omega_1-\omega_2)s}{(s+\omega_1)(s+\omega_2)}\right]V_i(s) \\
&= -\frac{\alpha s}{(s+\omega_1)(s+\omega_2)}V_i(s), \qquad \alpha = \frac{R_5(\omega_2-\omega_1)}{R_4} \qquad \text{(+1 point)}
\end{aligned}$$

(b) (4 points) Compute α, ω_1 and ω_2 in terms of the values of the circuit elements and select all resistors, capacitors and inductors such that

$$\alpha = \omega_2, \qquad \omega_1 = 2000, \qquad \omega_2 = 4000.$$

Sketch the magnitude of the frequency response of the circuit. Classify the circuit as low-pass, high-pass or band-pass.

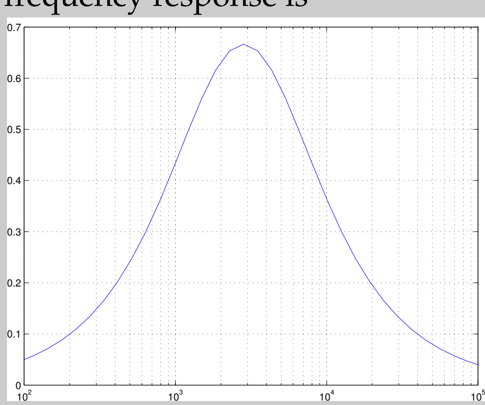
Solution:
From part (a):

$$\omega_1 = \frac{1}{R_1C_1}, \qquad \omega_2 = \frac{R_2}{L_2}, \qquad \alpha = \frac{R_5(\omega_2-\omega_1)}{R_4}.$$

For example, with $C_1 = 10^{-9}$ F, $L_2 = 1$ H, and $R_5 = 10^6\ \Omega$ we have

$$\begin{aligned}
R_1 &= \frac{1}{\omega_1 C_1} = \frac{1}{2}\times 10^6\ \Omega, \\
R_2 &= L_2\omega_2 = 4\times 10^3\ \Omega, \qquad (2)\\
R_4 &= \frac{R_5(\omega_2-\omega_1)}{\omega_2} = R_5 = 10^6\ \Omega. \qquad \text{(+2 points)}
\end{aligned}$$

The magnitude of the frequency response is



(+1 point)

This is a band-pass filter. (+1 point)

(c) (3 points) Use what you know about frequency response to compute the steady-state response of the above circuit to the inputs $v_1(t) = 3\cos(1000t)$ and $v_1(t) = -2\cos(3000t)$.

Hint: make sure it agrees with your answer to part (b).

Solution:

Using the frequency response method, we know that given an input of the form $v_i = A\cos(\omega t)$, the output of a linear circuit will be

$$v_o^{SS}(t) = A|T(j\omega)|\cos(\omega t + \angle T(j\omega)) \qquad \text{(+1 point)}$$

For

$$T(s) = \frac{V_o(s)}{V_i(s)} = -\frac{\alpha s}{(s+\omega_1)(s+\omega_2)}$$

we have

$$\begin{aligned} |T(j\omega)| &= \frac{|\alpha\omega|}{\sqrt{\omega^2+\omega_1^2}\sqrt{\omega^2+\omega_2^2}} \\ &= \frac{|4\times 10^3\omega|}{\sqrt{\omega^2+4\times 10^6}\sqrt{\omega^2+16\times 10^6}} \\ &= \frac{|4\omega/10^3|}{\sqrt{\omega^2/10^6+4}\sqrt{\omega^2/10^6+16}} \end{aligned}$$

and

$$\begin{aligned} \angle T(j\omega) &= \angle -j\alpha\omega - \angle j\omega+\omega_1 - \angle j\omega+\omega_2 \\ &= -\pi/2 - \tan^{-1}\omega/\omega_1 - \tan^{-1}\omega/\omega_2 \end{aligned} \qquad \text{(+0.5 point)}$$

Evaluating the magnitudes

$$\begin{aligned} |T(j1000)| &= \frac{4}{\sqrt{1+4}\sqrt{1+16}} = \frac{4}{\sqrt{85}} \approx 0.43 \\ |T(j3000)| &= \frac{4\times 3}{\sqrt{9+4}\sqrt{9+16}} = \frac{12}{\sqrt{325}} \approx 0.66 \end{aligned} \qquad \text{(+0.5 point)}$$

and the phases

$$\begin{aligned} \angle T(j1000) &= -\pi/2 - \tan^{-1}1/2 - \tan^{-1}1/4 \approx -2.28 \\ \angle T(j3000) &= -\pi/2 - \tan^{-1}3/2 - \tan^{-1}3/4 \approx -3.20 \end{aligned} \qquad \text{(+0.5 point)}$$

Therefore

$$\begin{aligned} v_2^{SS}(t) &\approx 3\times 0.43\cos(1000t-2.28) = 1.29\cos(1000t-2.28), \\ v_2^{SS}(t) &\approx -2\times 0.66\cos(3000t-3.20) = -1.32\cos(3000t-3.20) \end{aligned} \qquad \text{(+0.5 point)}$$

(d) (4 points) Design a circuit with the same transfer function as the one in parts (a-b) that uses at most two Op-Amps and no inductors.

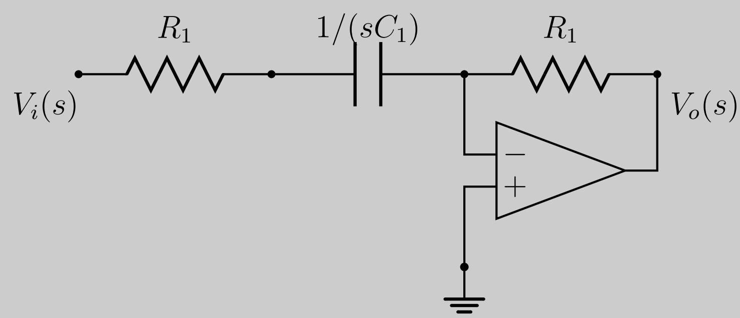
Solution:
One solution is to use a cascade realization

$$T(s) = \frac{\omega_2 s}{(s+\omega_1)(s+\omega_2)} = T_1(s) \times T_2(s)$$

where

$$T_1(s) = -\frac{s}{s+\omega_1}, \qquad T_2(s) = -\frac{\omega_2}{s+\omega_2} \qquad \text{(+1 point)}$$

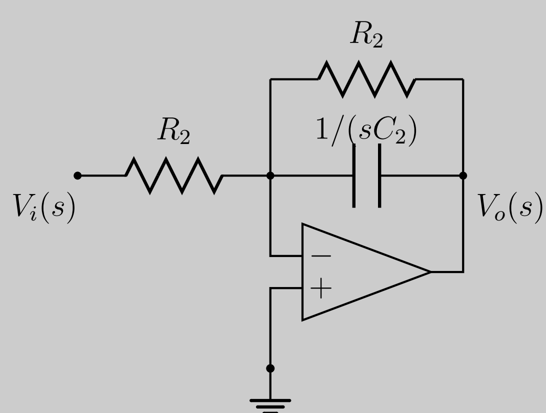
We can use the inverter OpAmp realization for $T_1(s)$



where

$$T_1(s) = -\frac{R_1}{R_1 + \frac{1}{sC_1}} = -\frac{s}{s+\omega_1}, \qquad \omega_1 = \frac{1}{R_1 C_1} \qquad \text{(+1 point)}$$

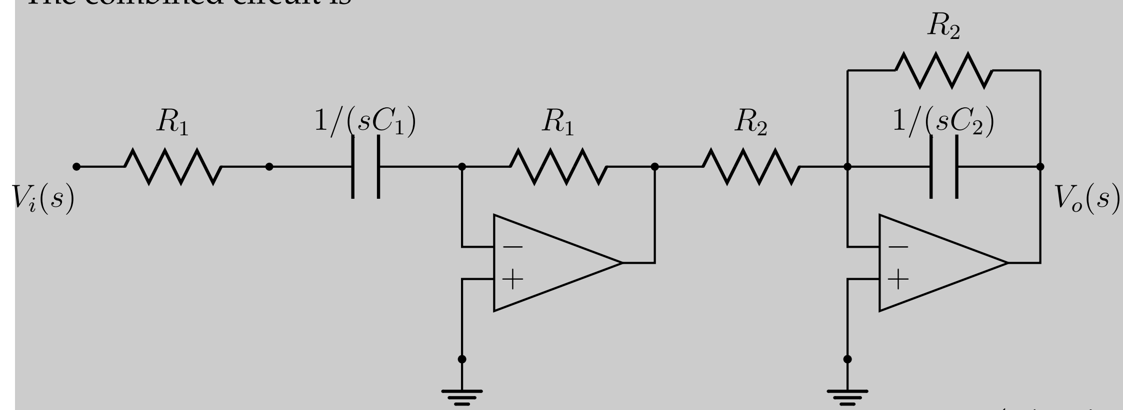
and the inverter OpAmp realization for $T_2(s)$



where

$$T_2(s) = -\frac{1}{R_2}\frac{1}{sC_2 + \frac{1}{R_2}} = -\frac{\omega_2}{s+\omega_2}, \qquad \omega_2 = \frac{1}{R_2 C_2} \qquad \text{(+1 point)}$$

The combined circuit is



(+1 point)

where with $C_1 = C_2 = 10^{-9}$ F we have

$$R_1 = \frac{1}{\omega_1 C_1} = \frac{1}{2} \times 10^6\ \Omega,$$
$$R_2 = \frac{1}{\omega_2 C_2} = \frac{1}{4} \times 10^6\ \Omega.$$

(e) (1 point (bonus)) Redesign the circuit you obtained in part (d) to use at most one Op-Amp and no inductors. If your design in part (d) already satisfies these requirements then you automatically get the bonus point.

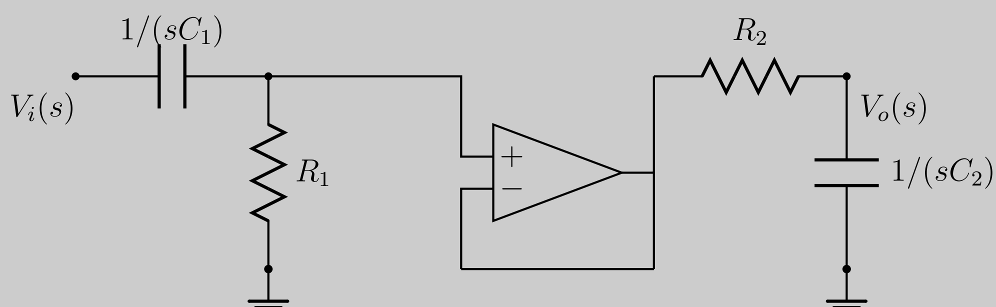
Solution:
One solution is to use a cascade realization

$$T(s) = \frac{\omega_2 s}{(s+\omega_1)(s+\omega_2)} = T_1(s) \times T_2(s)$$

where

$$T_1(s) = \frac{s}{s+\omega_1}, \qquad T_2(s) = \frac{\omega_2}{s+\omega_2}$$

connected by a non-inverting buffer OpAmp module.



(+1 point)

With $C_1 = C_2 = 10^{-9}$ F we have

$$R_1 = \frac{1}{\omega_1 C_1} = \frac{1}{2} \times 10^6\ \Omega,$$
$$R_2 = \frac{1}{\omega_2 C_2} = \frac{1}{4} \times 10^6\ \Omega.$$

Total for Question 3: 16

Total for Question 3: 1 (bonus)