MAE140 – Linear Circuits – Winter – 2014
Final, March 18th

Instructions

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.
2. You have 170 minutes.
3. Do not forget to write your name and student number.
4. This exam has 5 questions, for a total of 50 points and 6 bonus points.

1. **Dynamic OpAmp Circuit**

   (a) (1 point) Calculate that the gain K from v_s to v_0 in the OpAmp circuit of Fig. 1a.

   Solution: This is a standard non-inverting OpAmp circuit for which $K = \frac{R_1+R_2}{R_2}$. (+1 point)

   (b) (2 points) The circuit in Fig. 1b is an improved model that takes into account the OpAmp's dynamic response. Substitute this model for the OpAmp in the circuit of Fig. 1a to show that:

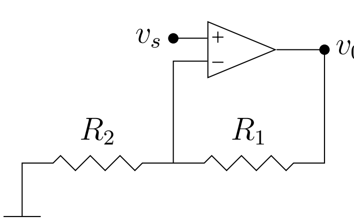
$$V_0(s) = \frac{\mu_0 K}{K + \mu_0} \frac{\omega_c(K)}{s + \omega_c(K)} V_s(s)$$

   where

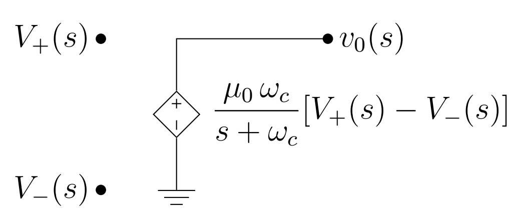
$$\omega_c(K) = \left(1 + \frac{\mu_0}{K}\right) \omega_c$$

   Solution: Substitute Fig. 1b into Fig. 1a. (+1 point)
   The voltage $V_-(s)$ can be obtained by voltage-division:

$$V_-(s) = \frac{R_2}{R_1 + R_2} V_0(s) \qquad \text{(+0.5 point)}$$



(a)



(b)

Figure 1: OpAmp circuit

This is

$$V_0(s) = KV_-(s)$$

Using the OpAmp model:

$$V_-(s) = \frac{1}{K}\frac{\mu_0\omega_c}{s+\omega_c}\left[V_+(s) - V_-(s)\right]$$

from which

$$\left(1 + \frac{1}{K}\frac{\mu_0\omega_c}{s+\omega_c}\right)V_-(s) = \frac{1}{K}\frac{\mu_0\omega_c}{s+\omega_c}V_s(s) \qquad \text{(+0.5 point)}$$

Cleaning up

$$(s+\omega_c(K))V_-(s) = \left(s + \omega_c + \frac{\mu_0}{K}\omega_c\right)V_-(s) = \frac{\mu_0\omega_c}{K}V_s(s)$$

and

$$V_0(s) = KV_-(s) = \frac{\mu_0\omega_c}{s+\omega_c(K)}V_s(s) = \frac{\mu_0}{K+\mu_0}\frac{\omega_c(K)}{s+\omega_c(K)}V_s(s) \qquad \text{(+1 point)}$$

(c) (3 points) What type of filter response does the circuit has when $\mu_0 > 0$? Calculate the filter gain and cutoff frequency. Is this circuit stable?

Solution: The first order circuit with

$$T(s) = \frac{V_0(s)}{V_s(s)} = \frac{\mu_0 K}{K+\mu_0}\frac{\omega_c(K)}{s+\omega_c(K)}$$

is a *low-pass filter*. (+1 point)

The cutoff frequency is at the pole

$$\omega_c(K) \qquad \text{(+0.5 point)}$$

Gain is at $s = 0$

$$K = |T(0)| = \frac{\mu_0 K}{K+\mu_0}. \qquad \text{(+0.5 point)}$$

It is a stable circuit since its pole is at

$$s = -\omega_c(K) = -\left(1 + \frac{\mu_0}{K}\right)\omega_c < 0 \qquad \text{(+1 point)}$$

(d) (2 points) Explain what happens when $\mu_0 \gg K$. Is this circuit stable?

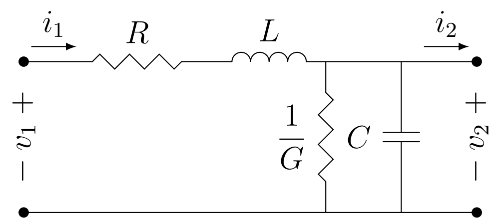


Figure 2: Circuit for Question 2

Solution: With $\mu_0 \gg K$ cutoff frequency is

$$\omega_c(K) = \left(1 + \frac{\mu_0}{K}\right)\omega_c \text{ is much larger than } \omega_c \qquad \text{(+0.5 point)}$$

Gain

$$K = |T(0)| = \frac{\mu_0 K}{K + \mu_0} = \frac{K}{\frac{K}{\mu_0} + 1} \approx K \qquad \text{(+1 point)}$$

which is the gain you designed for.
It is stable since $\mu_0 > 0$ and $-\omega_c(K) < 0$. (+0.5 point)

(e) (2 points) Explain what happens when $\mu_0 < 0$ and $|\mu_0| > K$. Is this circuit stable?

Solution: With $\mu_0 < 0$ and $|\mu_0| > K$ the pole is at

$$s = -\omega_c(K) = -\left(1 - \frac{|\mu_0|}{K}\right)\omega_c > 0 \qquad \text{(+1 point)}$$

and the circuit is not stable.
Voltages will grow out of bounds and OpAmp will reach saturation. (+1 point)

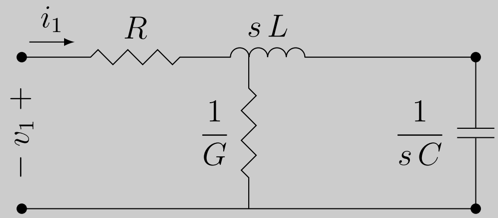
(f) (1 point (bonus)) What changes in the circuit correspond to $\mu_0 < 0$?

Solution:
Flipping $+$ and $-$ in the OpAmp.
(+1 bonus point)

2. **Circuit Equivalence**

(a) (2 points) Assuming zero initial conditions, find the impedance equivalent to the circuit in Figure 2 as seen from the terminals around v_1. The answer should be given as a ratio of two polynomials.

Solution:
Convert to the s-domain and leave v_2 open:



(+1 point)

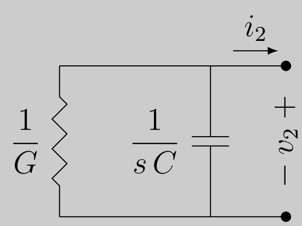
With terminal v_2 open the equivalent impedance is

$$
\begin{aligned}
Z(s) &= R + sL + \frac{1}{G + sC} \\
&= \frac{(R + sL)(G + sC) + 1}{G + sC}
\end{aligned}
$$

(+1 point)

(b) (2 points) Assuming zero initial conditions, find the impedance equivalent to the circuit in Figure 2 as seen from the terminals around v_2. The answer should be given as a ratio of two polynomials.

Solution:
Convert to the s-domain leaving v_1 open:



(+1 point)

With terminal v_1 open the equivalent impedance is

$$
Z(s) = \frac{1}{G + sC}
$$

(+1 point)

(c) (3 points) Show that

$$
\begin{aligned}
V_2(s) - V_1(s) + (R + s\,L)I_1(s) = 0, \\
I_2(s) - I_1(s) + (G + s\,C)V_2(s) = 0
\end{aligned}
$$

Solution:
Current $I_1(s)$ is

$$
I_1(s) = \frac{V_1(s) - V_2(s)}{R + sL}
$$

(+1 point)

from which the first equation follows.
Sum of the currents on capacitor and conductance is $I_1(s) - I_2(s)$. (+1 point)
Voltage $V_2(s)$ is

$$V_2(s) = \frac{1}{G + sC}[I_1(s) - I_2(s)] \qquad \text{(+1 point)}$$

from which the second equation follows

(d) (2 points (bonus)) Use part (c) to show that

$$Z_1(s) = \frac{V_1(s)}{I_1(s)} = \frac{(R + Ls) + [1 + (G + Cs)(R + Ls)]Z_2(s)}{1 + (G + Cs)Z_2(s)}, \qquad Z_2(s) = \frac{V_2(s)}{I_2(s)}.$$

Solution:
From the second equation

$$I_1(s) = (G + s\,C)V_2(s) + I_2(s) \qquad \text{(+1 point)}$$

From first equation:

$$\begin{aligned} V_1(s) &= (R + s\,L)I_1(s) + V_2(s) \\ &= (R + s\,L)[(G + s\,C)V_2(s) + I_2(s)] + V_2(s) \end{aligned} \qquad \text{(+1 point)}$$

Then

$$\begin{aligned} \frac{V_1(s)}{I_1(s)} &= \frac{(R + s\,L)[(G + s\,C)V_2(s) + I_2(s)] + V_2(s)}{(G + s\,C)V_2(s) + I_2(s)} \\ &= \frac{[1 + (R + s\,L)(G + s\,C)]V_2(s) + (R + L\,s)I_2(s)}{(G + s\,C)V_2(s) + I_2(s)} \\ &= \frac{[1 + (R + s\,L)(G + s\,C)]Z_2(s) + (R + L\,s)}{(G + s\,C)Z_2(s) + 1} \end{aligned} \qquad \text{(+1 point)}$$

(e) (3 points) Use part (d) to compute $Z_1(s)$ when $Z_2(s) = 0$ and $Z_2(s) \to \infty$. Explain whether it is possible or not to relate your answer to parts (a) and (b).

Solution:
When $Z_2(s) = 0$

$$Z_1(s) = (R + Ls) \qquad \text{(+1 point)}$$

This corresponds to a short-circuit in the terminal v_2 (+0.5 point)
When $Z_2(s) \to \infty$

$$Z_1(s) = \frac{1 + (R + s\,L)(G + s\,C)}{G + s\,C} \qquad \text{(+1 point)}$$

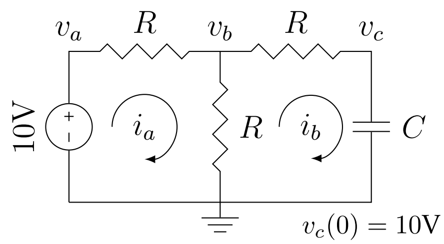


Figure 3: Circuit for Question 3

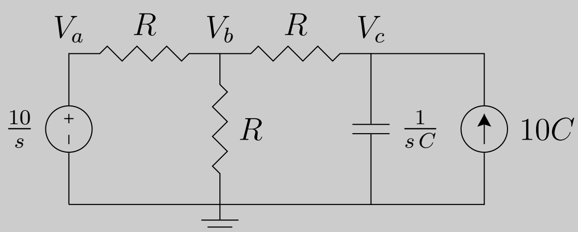
This corresponds to an open-circuit in the terminal v_2 and has been computed in part (a). (+0.5 point)

3. **Circuit Analysis**

(a) (4 points) Convert the circuit in Fig. 3 into the s-domain and formulate its node-voltage equations. Use the node-voltage, initial conditions and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns. Make sure your final equations only involve node-voltages.

Solution:

Convert to the s-domain taking into account the initial conditions



(+1 point)

Method #2 provides the following equations obtained by inspection:

$$V_a(s) = \frac{10}{s}$$

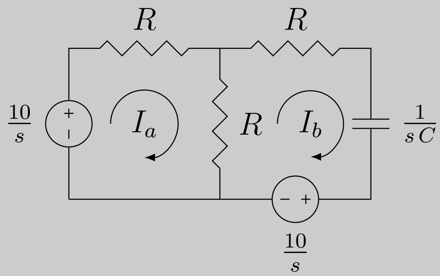
$$\begin{bmatrix} -\frac{1}{R} & \frac{3}{R} & -\frac{1}{R} \\ 0 & -\frac{1}{R} & \frac{1}{R} + sC \end{bmatrix} \begin{pmatrix} V_a(s) \\ V_b(s) \\ V_c(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 10C \end{pmatrix} \qquad \text{(+2 points)}$$

which should be solved for the unknowns $V_a(s)$, $V_b(s)$ and $V_c(s)$. (+1 point)

(b) (4 points) Convert the circuit in Fig. 3 into the s-domain and formulate its mesh-current equations. Use the mesh-currents, initial conditions and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns. Make sure your final equations only involve mesh-currents.

Solution:

Convert to the s-domain taking into account the initial conditions



(+1 point)

Writing the equations by inspection:

$$\begin{bmatrix} 2R & -R \\ -R & 2R + \frac{1}{sC} \end{bmatrix} \begin{pmatrix} I_a(s) \\ I_b(s) \end{pmatrix} = \begin{pmatrix} \frac{10}{s} \\ -\frac{10}{s} \end{pmatrix} \qquad \text{(+2 points)}$$

which should be solved for the unknowns $I_a(s)$, $I_b(s)$. (+1 point)

(c) (2 points (bonus)) Solve the equations in part (a) or part (b) to show that

$$v_c(t) = 5\left(1 + e^{-\frac{2}{3}\frac{1}{RC}t}\right), \qquad t \geq 0$$

Solution:

Solving the node-voltage equations:

$$\begin{aligned} \frac{3}{R}V_b(s) &= \frac{1}{R}\frac{10}{s} + \frac{1}{R}V_c(s) \\ \left(\frac{1}{R} + sC\right)V_c(s) &= 10C + \frac{1}{R}V_b(s) \\ &= 10C + \frac{1}{3R}\frac{10}{s} + \frac{1}{3R}V_c(s) \end{aligned}$$

from which

$$\left(s + \frac{2}{3RC}\right)V_c(s) = 10 + \frac{1}{3}\frac{10}{sRC} \qquad \text{(+1 point)}$$

and

$$V_c(s) = \frac{10}{s + \frac{2}{3RC}} + \frac{10}{3RC}\frac{1}{s(s + \frac{2}{3RC})} = \frac{k_1}{s} + \frac{k_2}{s + \frac{2}{3RC}}.$$

Calculating residues

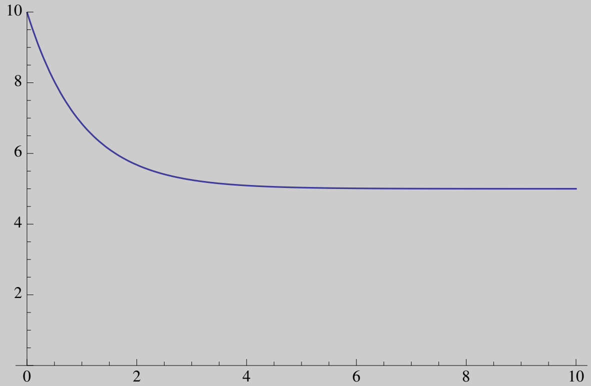
$$\begin{aligned} k_1 &= \lim_{s\to 0} \frac{10}{3RC}\frac{1}{s + \frac{2}{3RC}} = \frac{10}{3RC}\frac{3RC}{2} = 5, \\ k_2 &= \lim_{s\to -\frac{2}{3RC}} 10 + \frac{10}{3sRC} = 10 - \frac{10}{2} = 5 \end{aligned}$$

The Laplace inverse is

$$v_c(t) = 5 + 5e^{-\frac{2}{3RC}t}, \qquad t \geq 0 \qquad \text{(+1 point)}$$

(d) (2 points) Sketch the voltage $v_c(t)$ obtained in part (c) and explain how one could show that $\lim_{t\to\infty} v_c(t) = 5\text{V}$ without solving the equations in part (a) or (b).

Solution:
A sketch of the solution should look like:



(+1 point)

Recall that all sources are constant and the capacitor will achieve steady-state only if its current is zero. The resulting circuit is a voltage divider from which

$$v_c^{ss} = v_b^{ss} = \frac{R}{R+R}10\text{V} = 5\text{V}. \qquad \text{(+1 point)}$$

4. **OpAmp Circuit Analysis and Design**

(a) (3 points) Assume zero initial conditions, convert the circuit in Fig. 4a to the s-domain then show that the transfer-function from $V_s(s)$ to $V_0(s)$ is equal to

$$T(s) = \frac{V_0(s)}{V_s(s)} = -K\frac{s+\omega_1}{s+\omega_2}, \qquad K = \frac{R_1}{R_2}, \qquad \omega_1 = \frac{1}{R_1C_1}, \qquad \omega_2 = \frac{1}{R_2C_2}$$

Solution:
Convert to the s-domain

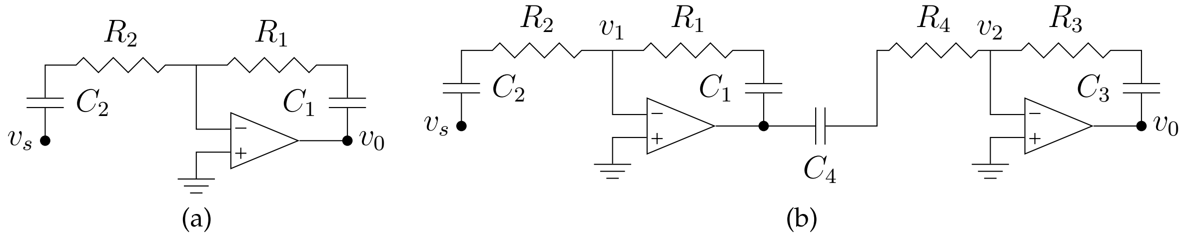
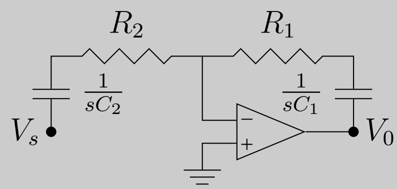


Figure 4: Circuits for Questions 3 and 4



(+1 point)

Circuit is a standard inverting OpAmp circuit for which the transfer-function is

$$T(s) = -\frac{Z_1(s)}{Z_2(s)}, \qquad Z_1(s) = R_1 + \frac{1}{sC_1}, \qquad Z_2(s) = R_2 + \frac{1}{sC_2} \qquad \text{(+2 points)}$$

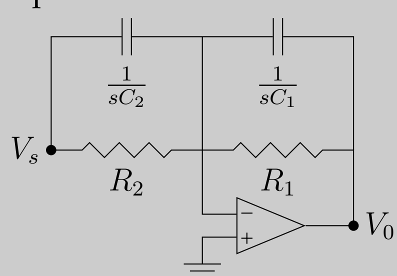
which can be put in the form

$$T(s) = -\frac{R_1}{R_2}\frac{s + \frac{1}{R_1C_1}}{s + \frac{1}{R_2C_2}} \qquad \text{(+1 point)}$$

(b) (2 points) Design a different circuit that has the exact same transfer-function as the circuit in Fig. 4a. Clearly show the relationship between the components in your circuit and K, ω_1 and ω_2.

Solution:

A possible solution is:



(+1 point)

for which

$$Z_1(s) = \frac{1}{\frac{1}{R_1} + sC_1}, \qquad Z_2(s) = \frac{1}{\frac{1}{R_2} + sC_2} \qquad \text{(+0.5 point)}$$

and

$$T(s) = -\frac{Z_1(s)}{Z_2(s)} = -\frac{sC_2 + \frac{1}{R_2}}{sC_1 + \frac{1}{R_1}} = -K\frac{s+\omega_1}{s+\omega_2} \qquad \text{(+0.5 point)}$$

with

$$K = \frac{C_2}{C_1}, \qquad \omega_1 = \frac{1}{R_2C_2}\omega_2 \qquad = \frac{1}{R_1C_1}.$$

(c) (2 points) Calculate the transfer-function from $V_s(s)$ to $V_0(s)$ for the circuit in Fig. 4b. Clearly explain your reasoning.

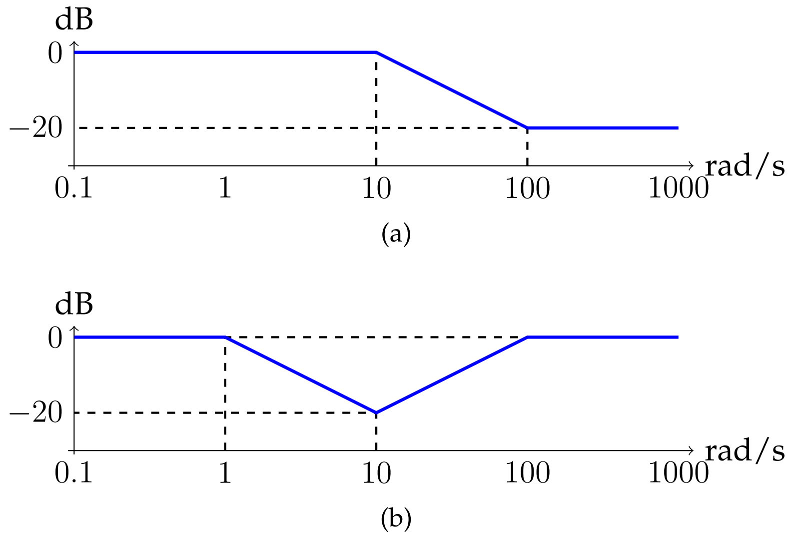


Figure 5: Bode diagrams for Question 4

Solution:
Because the circuits are connected at the output of the first OpAmp which has low impedance (ideal model is a voltage source) the voltage at that point does not depend on the second circuit. (+1 point)

Therefore

$$T(s) = T_1(s)T_2(s) = K\frac{(s+\omega_1)(s+\omega_3)}{(s+\omega_2)(s+\omega_4)}, \qquad \text{(+1 point)}$$

$$K = K_1K_2 = \frac{R_1R_3}{R_2R_4}, \qquad \omega_i = \frac{1}{R_iC_i}, \qquad i = 1, \ldots, 4.$$

(d) (3 points) A student built the circuit in Fig. 4b in the lab and observed that when $R_1 = R_4$ and $C_1 = C_4$ he/she could short circuit the nodes labeled v_1 and v_2 and the circuit would continue to work as if nothing had happened. Is he/she correct? Why?

5. **Frequency Response**

(a) (3 points) Calculate R_1, C_1, R_2 and C_2 so that the straight line approximation of the magnitude Bode diagram of the frequency response of the circuit in Fig. 4a is equal to the one shown in Fig. 5a. Explain your reasoning.

Solution:
From the approximate Bode plot there should be a pole at $s = -10$ and a zero at $s = -100$.

(+1 point)

The tranfer-function is of the form

$$T(s) = -K\frac{s+100}{s+10} \qquad \text{(+0.5 point)}$$

from which K can be determined at

$$0\text{dB} = 1 = |T(0)| = K\frac{100}{10} \qquad \Longrightarrow \qquad K = 0.1. \qquad \text{(+0.5 point)}$$

Values of components to satisfy those could be

$$R_1 = 10^6\ \Omega, \qquad R_2 = 10R_1 = 10^7\ \Omega, \qquad \text{(+1 point)}$$
$$C_1 = \frac{1}{\omega_1 R_1} = \frac{1}{100 \times 10^6} = 10^{-8}\text{F}, \qquad C_2 = \frac{1}{\omega_2 R_2} = \frac{1}{10 \times 10^7} = 10^{-8}\text{F}.$$

(b) (3 points) Use what you know about frequency response to estimate the amplitude of the steady-state response of the circuit in Fig. 4a when the components are chosen as in part (a) and $v_s(t) = 5\cos(t)$. Repeat for $v_s(t) = 2\cos(1000t + \pi/4)$?

Solution:

From the approximate Bode plot at $\omega = 1$ the gain is about 0 dB or 1. In this case the amplitude of the output should be approximately 5×1 V.

(+1.5 point)

From the Bode plot at $\omega = 1000$ the gain is about -20 dB or $10^{-1} = 0.1$. In this case the amplitude of the output should be approximately $2 \times 0.1 \approx 0.2$ V.

(+1.5 point)

(c) (1 point (bonus)) What about the phase of the steady-state responses in part (b)?

Solution:

$\omega = 1$ is before all poles and zeros contribute phase. The negative gain contributes 180° and so the phase of the output should be approximately 180°.

(+.5 point)

$\omega = 1000$ is well after all poles and zeros. The pole constributes -90° and the zero $+90^\circ$ a total net of 0. Again the negative gain contributes 180° and in this case the phase of the output should be approximately $45^\circ + 180^\circ$ since the source had a phase of $\pi/4$.

(+.5 point)

(d) (4 points) Calculate R_1, C_1, R_2, C_2, R_3, C_3, R_4, C_4 so that the straight line approximation of the magnitude Bode diagram of the frequency response of the circuit in Fig. 4b is equal to the one shown in Fig. 5b. Explain your reasoning.

Solution:
From the approximate Bode plot there should be a pole at $s = -1$, two zeros at $s = -10$ and one pole at $s = 100$.

(+1 point)

The tranfer-function is of the form

$$T(s) = -K\frac{(s+10)^2}{(s+1)(s+100)} \qquad \text{(+1 point)}$$

from which K can be determined at

$$0\text{dB} = 1 = |T(0)| = K\frac{100}{100} \qquad \Longrightarrow \qquad K = 1 \qquad \text{(+0.5 point)}$$

Values of components to satisfy those could be

$$R_1 = R_2 = R_3 = R_4 = 10^6\ \Omega, \qquad \text{(+1.5 point)}$$

$$C_1 = \frac{1}{\omega_1 R_1} = \frac{1}{10 \times 10^6} = 10^{-7}\text{F}, \qquad C_2 = \frac{1}{\omega_2 R_2} = \frac{1}{1 \times 10^6} = 10^{-6}\text{F}$$

$$C_3 = \frac{1}{\omega_3 R_3} = \frac{1}{10 \times 10^6} = 10^{-7}\text{F}, \qquad C_4 = \frac{1}{\omega_4 R_4} = \frac{1}{100 \times 10^6} = 10^{-8}\text{F}.$$