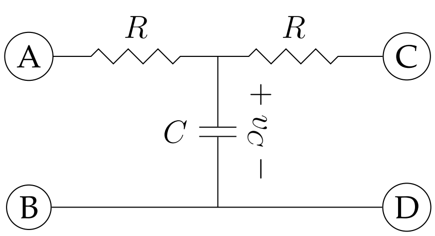
MAE140 – Linear Circuits – Winter – 2015
Final, March 20th

Instructions

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.
2. You have 170 minutes.
3. Do not forget to write your name and student number.
4. This exam has 6 questions, for a total of 60 points and 6 bonus points.



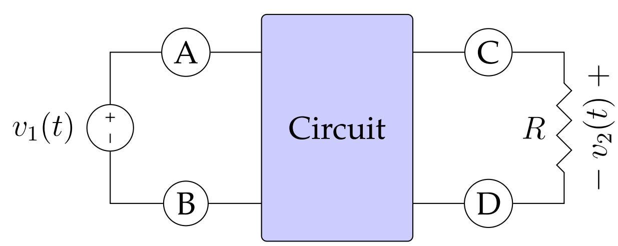


Figure 1: Circuits for Questions 1 and 2

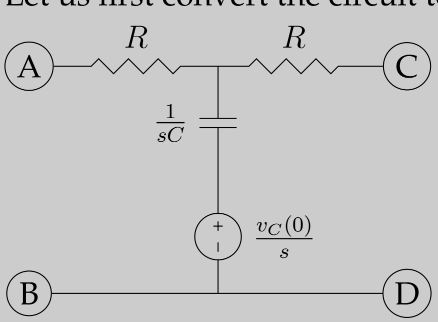
1. **Circuit Equivalence**

   IMPORTANT: All impedances should be given as a ratio of two polynomials!

   (a) (3 points) Do not assume zero-initial conditions and find the s-domain equivalent circuit as seen from terminals A and B in Fig. 1a.

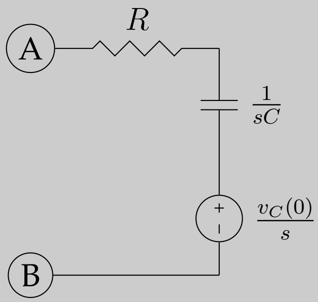
   Solution:

   Let us first convert the circuit to the s-domain to obtain:



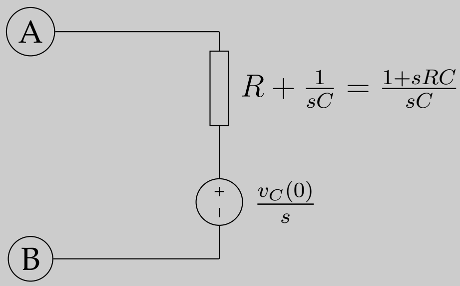
   (+1 point)

   As seen from A to B the circuit is simply:



(+1 point)

which is equivalent to:



(+1 point)

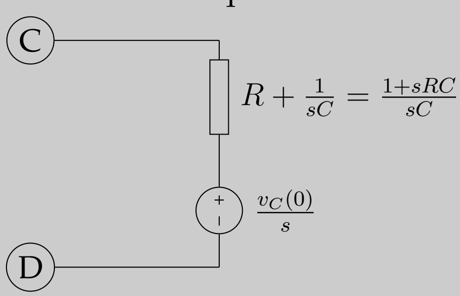
NOTE TO GRADERS: A Norton equivalent is also possible.

(b) (2 points) Do not assume zero-initial conditions and find the s-domain equivalent circuit as seen from terminals C and D in Fig. 1a.

Solution:

As shown in (a), the circuit is perfectly symmetric when seen from A to B or from C to D. (+1 point)

Therefore the equivalent circuit is:



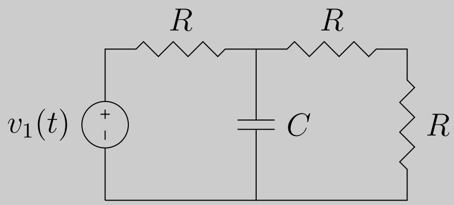
(+1 point)

(c) (5 points) Assume zero initial conditions and show that the transfer-function from $V_1(s)$ to $V_2(s)$ in Fig. 1b is equal to

$$T(s) = \frac{\omega_o}{2\,s + 3\,\omega_o}, \quad \omega_o = \frac{1}{RC} \tag{1}$$

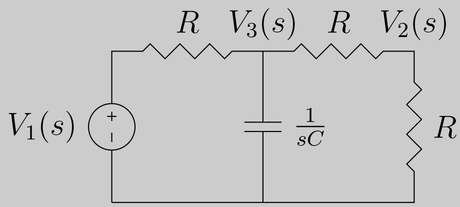
when the block labeled “Circuit” is the circuit in Fig. 1a.

Solution:
Substitute the circuit in Fig. 1a into the circuit in Fig. 1b to obtain:



(+1 point)

Assuming zero initial conditions and converting to the s-domain:



(+1 point)

Let us first calculate $V_3(s)$ using voltage division:

$$V_3(s) = \frac{Z(s)}{R + Z(s)} V_1(s) \qquad \text{(+0.5 point)}$$

where

$$Z(s) = \frac{1}{\frac{1}{2R} + sC} = \frac{2\frac{1}{C}}{\frac{1}{RC} + 2s} = \frac{2R\omega_o}{2s + \omega_o} \qquad \text{(+0.5 point)}$$

is the parallel association of the capacitor with the two R resistors in series. After some algebra:

$$\begin{aligned} V_3(s) &= \frac{Z(s)}{R + Z(s)} V_1(s) \\ &= \frac{\frac{2R\omega_o}{2s+\omega_o}}{R + \frac{2R\omega_o}{2s+\omega_o}} V_1(s) \\ &= \frac{2\omega_o}{2s + \omega_o + 2\omega_o} V_1(s) \\ &= \frac{2\omega_o}{2s + 3\omega_o} V_1(s) \qquad \text{(+0.5 point)} \end{aligned}$$

From $V_3(s)$, the voltage $V_2(s)$ is obtained again by voltage division:

$$V_3(s) = \frac{R}{R + R} V_2(s) = \frac{1}{2} V_2(s) \qquad \text{(+0.5 point)}$$

from which

$$V_3(s) = \frac{\omega_o}{2s + 3\omega_o} V_1(s) \qquad \text{(+0.5 point)}$$

and the transfer-function:

$$T(s) = \frac{V_3(v)}{V_1(s)} = \frac{\omega_o}{2s + 3\omega_o}. \qquad \text{(+0.5 point)}$$

2. **Frequency Response**

Consider the circuit in Fig. 1b where "Circuit" is the circuit in Fig. 1a for which you calculated the transfer-function $T(s) = V_2(s)/V_1(s)$ given in (1) in P.1.

IMPORTANT: In this question set $RC = 3/20$s/rad.

(a) (3 points) Calculate $|T(j\omega)|$ and $\angle T(j\omega)$.

Solution:
With $RC = 3/20$s/rad, $\omega_o = 20/3$ and from (1):

$$T(j\omega) = \frac{\omega_o}{2j\omega + 3\omega_o} = \frac{20/3}{2j\omega + 20} = \frac{10/3}{j\omega + 10} \qquad \text{(+1 point)}$$

Calculating the magnitude

$$\begin{aligned} |T(j\omega)| &= \frac{|10/3|}{|j\omega + 10|} \qquad \text{(+1 point)} \\ &= \frac{10/3}{\sqrt{\omega^2 + 100}} \end{aligned}$$

Calculating the phase

$$\begin{aligned} \angle T(j\omega) &= \angle \frac{10/3}{j\omega + 10} \\ &= \angle 10/3 - \angle j\omega + 10 \\ &= -\arctan \frac{\omega}{10}. \qquad \text{(+1 point)} \end{aligned}$$

(b) (3 points) Evaluate $|T(j\omega)|$ and $\angle T(j\omega)$ at

$$\omega = 0, \qquad \omega = 10\text{rad/s}, \qquad \omega = 100\text{rad/s}.$$

Approximate answers are fine.

Solution:
At $\omega = 0$

$$|T(j\omega)| = \frac{10/3}{10} = 0.33, \qquad \angle T(j\omega) = 0. \qquad \text{(+1 point)}$$

At $\omega = 10$

$$|T(j\omega)| = \frac{10/3}{10\sqrt{2}} \approx 0.2, \qquad \angle T(j\omega) = -\arctan 1 = -45°. \qquad \text{(+1 point)}$$

At $\omega = 100$

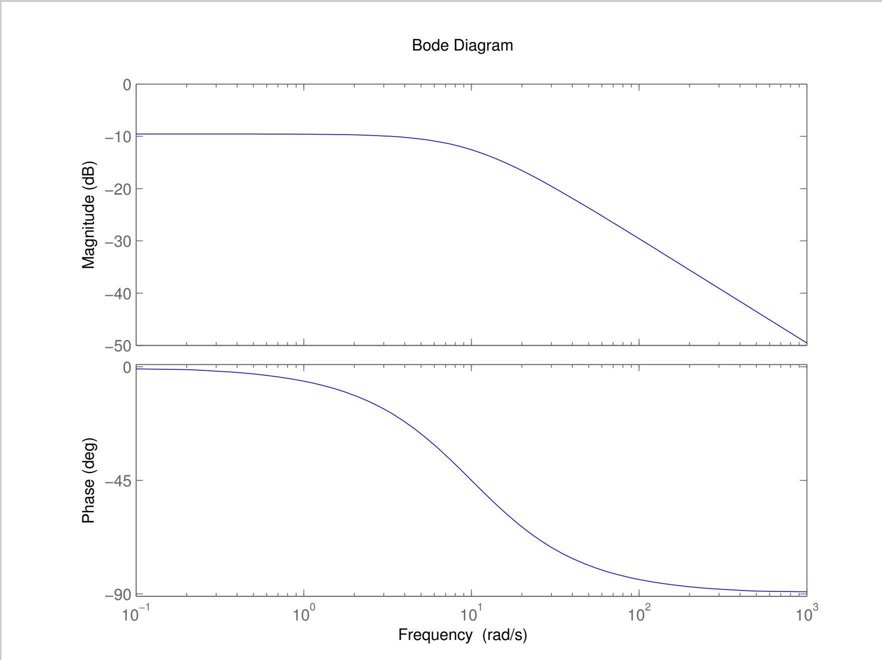
$$|T(j\omega)| = \frac{10/3}{\sqrt{10000 + 100}} \approx \frac{10/3}{100} = 0.03, \qquad \angle T(j\omega) = -\arctan 10 \approx -84°.$$

(+1 point)

(c) (3 points) Use the values you computed in (b) to sketch the plots of $|T(j\omega)|$ and $\angle T(j\omega)$ versus ω and classify the type of filter implemented by the circuit.

Solution:

A sketch should resemble the Bode plot of $T(s)$ which is:



(+2 point)

NOTE TO GRADERS: linear scale, rough sketches are fine as well.

The circuit is a low-pass filter. (+1 point)

(d) (1 point) Calculate the filter gain and cut-off frequency.

Solution:

The gain is $K = T(j0) = 1/3 \approx 0.33$ (+0.5 point)

The cut-off frequency is equal to the pole $\omega_c = 10\text{rad/s}$ (+0.5 point)

(e) (2 points (bonus)) Calculate the steady-state component of the output voltage v_2 when the input voltage v_1 is $v_1(t) = \sin(\omega t)$, $t \geq 0$, for $\omega = 10\text{rad/s}$.

Solution:

The key is to realize that

$$v_1(t) = \sin(\omega t) = \cos(\omega t - \pi/2) \qquad \text{(+1 point)}$$

and then use the frequency response method to calculate the steady-state solution:

$$\begin{aligned} v_2(t) &= |T(j10)| \cos(10t - \pi/2 + \angle T(j10)) \\ &\approx 0.2 \cos(10t - \pi/2 - \pi/4) \\ &= 0.2 \cos(10t - 3\pi/4). \end{aligned} \qquad \text{(+1 point)}$$

3. **Circuit Puzzle**

(a) (2 points) Calculate the Laplace transform of the input voltage:

$$v_1(t) = \frac{1}{2}\left(1 + e^{-2t}\right), \quad t \geq 0.$$

Solution:
Using linearity:

$$V_1(s) = \frac{1}{2}\left(\mathcal{L}\{1\} + \mathcal{L}\{e^{-2t}\}\right) \qquad \text{(+1 point)}$$

Using a table:

$$\begin{aligned} V_1(s) &= \frac{1}{2}\left(\frac{1}{s} + \frac{1}{s+2}\right) \\ &= \frac{1}{2}\frac{s+2+s}{s(s+2)} \\ &= \frac{(s+1)}{s(s+2)} \qquad \text{(+1 point)} \end{aligned}$$

NOTE TO GRADERS: Any correct answer is fine. Does not need to be factored in any special way.

(b) (4 points) When the voltage $v_1(t)$ from part (a) is applied to a linear circuit with zero initial conditions, the circuit produces the output:

$$v_2(t) = 1, \quad t \geq 0.$$

Calculate the transfer-function of this circuit.

Solution:
The Laplace transform of the output is:

$$V_2(s) = \mathcal{L}\{1\} = \frac{1}{s} \qquad \text{(+1 point)}$$

The circuit transfer-function must therefore equal:

$$T(s) = \frac{V_2(s)}{V_1(s)} \qquad \text{(+1 point)}$$

Plugging it together:

$$\begin{aligned} T(s) &= \frac{\frac{1}{s}}{\frac{(s+1)}{s(s+2)}} \qquad \text{(+1 point)} \\ &= \frac{s+2}{s+1} \qquad \text{(+1 point)} \end{aligned}$$

(c) (4 points) Explain why with most other input voltages of the form

$$v_1(t) = a + b\,e^{-2t}, \quad t \geq 0,$$

the output voltage of the circuit you found in (b) is always of the form

$$v_2(t) = \alpha + \beta\,e^{-t}, \quad t \geq 0.$$

Solution:
The Laplace transform of the input $v_1(t)$ is:

$$\begin{aligned} V_1(s) &= \frac{a}{s} + \frac{b}{s+2} \\ &= \frac{a(s+2) + bs}{s(s+2)} \\ &= \frac{(a+b)s + 2a}{s(s+2)}. \end{aligned} \qquad \text{(+1 point)}$$

When applied to the circuit it produces an output with transform:

$$\begin{aligned} V_2(s) &= T(s)V_1(s) \\ &= \frac{s+2}{s+1} \times \frac{(a+b)s + 2a}{s(s+2)} \\ &= \frac{(a+b)s + 2a}{s(s+1)}. \end{aligned} \qquad \text{(+1 point)}$$

Expanding in partial fractions:

$$\begin{aligned} V_2(s) &= \frac{(a+b)s + 2a}{s(s+1)} \qquad \text{(+1 point)} \\ &= \frac{\alpha}{s} + \frac{\beta}{s+1} \end{aligned}$$

from which

$$\begin{aligned} v_2(t) &= \mathcal{L}^{-1}\left\{\frac{\alpha}{s} + \frac{\beta}{s+1}\right\} \qquad \text{(+1 point)} \\ &= \alpha + \beta e^{-t}, \quad t \geq 0. \end{aligned}$$

NOTE TO READERS: There is no need to calculate α and β to answer this question!

(d) (2 points (bonus)) In part (c), is it possible to choose parameters a and b so that the constant in the output voltage $v_2(t)$ is zero, i.e. $\alpha = 0$? Explain what must occur in the frequency domain if this is to happen.

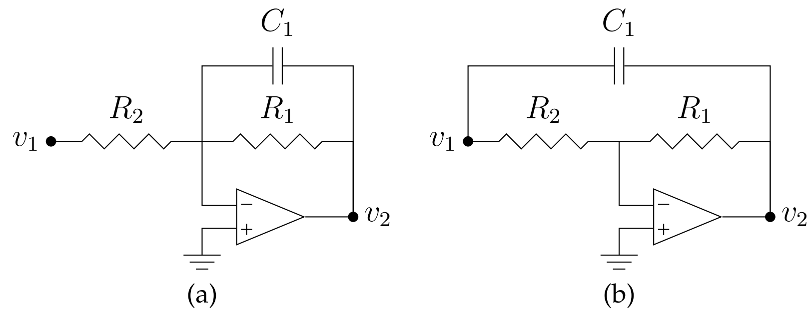


Figure 2: Circuits for Questions 4

Solution:
Calculate α and β using the residue formula:

$$\alpha = \lim_{s\to 0} \frac{(a+b)s+2a}{s+1} = 2a,$$
$$\beta = \lim_{s\to -1} \frac{(a+b)s+2a}{s} = b-a$$

It is possible to have $\alpha = 2a = 0$ if $a = 0$. In this case $\beta = b$. (+1 point)
Because the transform of the input $v_1(t)$ has a pole at $s = 0$ a pole zero cancellation must occur if a constant is not to be present at the output. Since

$$T(s) = \frac{s+2}{s+1}$$

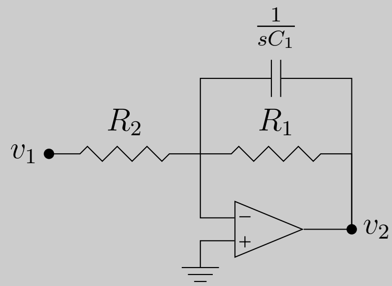
does not have a zero at $s = 0$ the only way this can happen is if $a = 0$, that is, if the input itself does not have a constant component. (+1 point)

4. **OpAmp Circuit Analysis**

(a) (3 points) Assume zero initial conditions, convert the circuit in Fig. 2a to the s-domain and show that the transfer-function from v_1 to v_2 is:

$$T(s) = -\frac{\frac{1}{R_2C_1}}{s+\frac{1}{R_1C_1}}$$

Solution:
Assume zero initial conditions and convert to the s-domain:



(+1 point)

The circuit is a standard inverting OpAmp circuit for which the transfer-function is

$$T(s) = -\frac{Z_1(s)}{Z_2(s)}, \qquad Z_1(s) = \frac{1}{\frac{1}{R_1} + sC_1}, \qquad Z_2(s) = R_2 \qquad \text{(+1 point)}$$

which can be put in the form

$$T(s) = -\frac{1}{R_2} \times \frac{1}{sC_1 + \frac{1}{R_1}} = -\frac{1}{R_2 C_1} \times \frac{1}{s + \frac{1}{R_1 C_1}}. \qquad \text{(+1 point)}$$

(b) (2 points) Calculate the circuit's gain and cut-off frequency.

Solution:
Because the circuit is a low-pass filter the gain is

$$|T(j0)| = \frac{1}{R_2 C_1} \times \frac{1}{\frac{1}{R_1 C_1}} = \frac{R_1}{R_2} \qquad \text{(+1 point)}$$

and the cutoff frequency is the pole $\omega_c = \frac{1}{R_1 C_1}$. (+1 point)

(c) (3 points) Show how R_1, R_2 and C_1 can be selected in the circuit in Fig. 2a so that the magnitude of the frequency response of the circuit in Fig. 2a be exactly the same as the magnitude of the frequency response of the circuit in Fig. 1 from P.1(c) and P.2, for which $T(s) = V_2(s)/V_1(s)$ is (1). Name one important difference between these two circuits.

Solution:
The frequency response of $T(s)$ from P.1(c) and P.2 is

$$T(j\omega) = \frac{10/3}{j\omega + 10}$$

The frequency response of $T(s)$ in P.4 is

$$T(j\omega) = -\frac{1}{R_2 C_1} \times \frac{1}{j\omega + \frac{1}{R_1 C_1}} \qquad \text{(+1 point)}$$

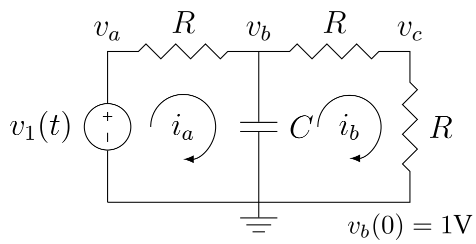


Figure 3: Circuit for Question 5

Their magnitudes will be the same if

$$\frac{1}{R_2C_1} = \frac{10}{3}, \qquad \frac{1}{R_1C_1} = 10. \qquad \text{(+1 point)}$$

This is clearly possible since there are three free parameters and only two constraints. For example:

$$C_1 = 10^{-7}\text{F}, \qquad R_1 = 10^6, \qquad R_2 = 3 \times 10^6\Omega.$$

NOTE TO GRADER: answer is correct if they indicate how it is possible without calculating R_1, R_2 and C_1.

An important difference is that the OpAmp circuit can be connected with other circuits without changing its transfer-function. It can also have gains higher than one. (+1 point)

NOTE TO GRADER: any correct advantage earns a point.

(d) (2 points) A student misconnected the capacitor C_1 resulting in the circuit in Fig. 2b. Calculate the transfer-function from v_1 to v_2 in Fig. 2b and explain the differences between this circuit and the one in Fig. 2a.

Solution:

The capacitor C_1 does not affect the transfer-function in the circuit in Fig. 2b because the circuit from v_1 to v_2 is a standard inverting OpAmp. Its transfer-function is:

$$\frac{v_2}{v_1} = -\frac{R}{R} = -1. \qquad \text{(+1 point)}$$

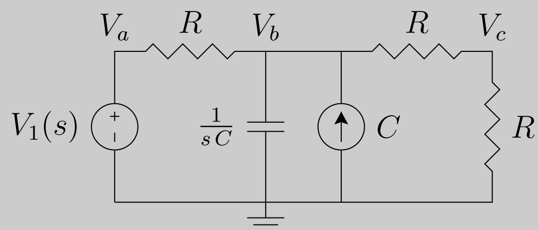
so the circuit in Fig. 2b is simply an amplifier whereas the circuit in Fig. 2a is a low-pass filter. (+1 point)

5. **Circuit Analysis**

(a) (4 points) Convert the circuit in Fig. 3 into the s-domain and formulate its node-voltage equations. Use the node-voltage, initial conditions and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns. Make sure your final equations only involve node-voltages.

Solution:

Convert to the s-domain taking into account the initial conditions



(+1 point)

Method #2 provides the following equations obtained by inspection:

$$V_a(s) = V_1(s)$$

$$\begin{bmatrix} -\frac{1}{R} & sC + \frac{2}{R} & -\frac{1}{R} \\ 0 & -\frac{1}{R} & \frac{2}{R} \end{bmatrix} \begin{pmatrix} V_a(s) \\ V_b(s) \\ V_c(s) \end{pmatrix} = \begin{pmatrix} C \\ 0 \end{pmatrix}$$

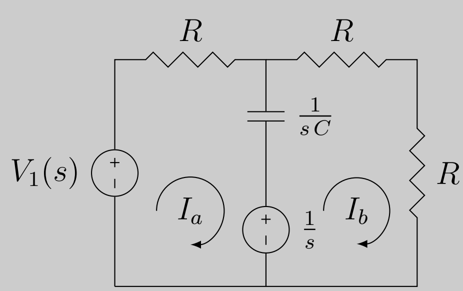
(+2 points)

which should be solved for the unknowns $V_a(s)$, $V_b(s)$ and $V_c(s)$. (+1 point)

(b) (4 points) Convert the circuit in Fig. 3 into the s-domain and formulate its mesh-current equations. Use the mesh-currents, initial conditions and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns. Make sure your final equations only involve mesh-currents.

Solution:

Convert to the s-domain taking into account the initial conditions



(+1 point)

Writing the equations by inspection:

$$\begin{bmatrix} R + \frac{1}{sC} & -\frac{1}{sC} \\ -\frac{1}{sC} & 2R + \frac{1}{sC} \end{bmatrix} \begin{pmatrix} I_a(s) \\ I_b(s) \end{pmatrix} = \begin{pmatrix} V_1(s) - \frac{1}{s} \\ \frac{1}{s} \end{pmatrix}$$

(+2 points)

which should be solved for the unknowns $I_a(s)$, $I_b(s)$. (+1 point)

(c) (2 points) After noticing that the circuit in Fig. 3 is the same as the one considered in P.1(c) and P.2, use the node-voltage equations from (a) or the mesh-current equations from (b) to verify the formula obtained for $T(s)$ in (1).
Hint: You can assume zero initial conditions for this part!

Solution:
We will solve for $V_2(s) = V_c(s)$ using node-voltage equations. Note that the third equation is:

$$\frac{2}{R}V_c(s) = \frac{1}{R}V_b(s) \qquad \Longrightarrow \qquad V_b(s) = 2V_c(s) \qquad \text{(+1 point)}$$

Substituting and solving the second equation:

$$-\frac{1}{R}V_1(s) + 2\left(sC + \frac{2}{R}\right)V_c(s) - \frac{1}{R}V_c(s) = 0$$

we obtain after dividing by C on both sides

$$\left(2s + 3\frac{1}{RC}\right)V_c(s) = \frac{1}{RC}V_1(s)$$

which is equivalent to $T(s)$. (+1 point)

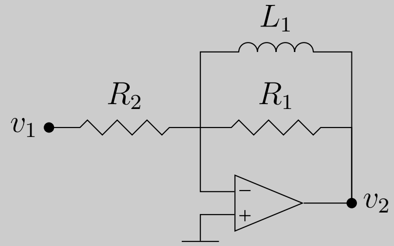
6. **Circuit Transformations**

A friend told you that if you pick any linear circuit and replace every capacitor by an inductor and vice-versa you will "flip" your circuit. For example, a low-pass filter becomes a high-pass filter, and vice-versa.

(a) (4 points) Verify your friend's claim for the circuit in Fig. 2a by showing that the transfer-function of the flipped circuit when the capacitor C_1 is replaced by an inductor L_1 is

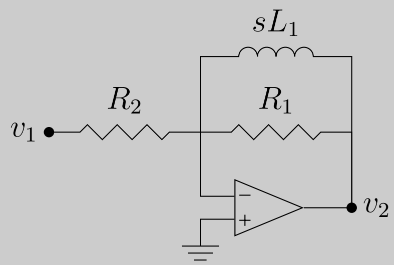
$$\tilde{T}(s) = -\frac{R_1}{R_2} \times \frac{s}{s + \frac{R_1}{L_1}}$$

Solution:
The flipped circuit is:



(+1 point)

Assuming zero initial conditions and converting to the s-domain:



(+1 point)

The circuit is a standard inverting OpAmp circuit with transfer-function:

$$T(s) = -\frac{Z_1(s)}{Z_2(s)}, \qquad Z_1(s) = \frac{1}{\frac{1}{R_1} + \frac{1}{sL_1}}, \qquad Z_2(s) = R_2 \qquad \text{(+1 point)}$$

which can be put in the form

$$\tilde{T}(s) = -\frac{1}{R_2} \times \frac{1}{\frac{1}{sL_1} + \frac{1}{R_1}} = -\frac{R_1}{R_2} \times \frac{s}{s + \frac{R_1}{L_1}}. \qquad \text{(+1 point)}$$

(b) (2 points) Verify that if you also flip the values of the inductors and capacitors, for example $C_1 = 1/L_1$, then

$$\tilde{T}(s) = T(1/s).$$

Solution:
We evaluate the transfer function

$$T(1/s) = -\frac{1}{R_2 C_1} \times \frac{1}{\frac{1}{s} + \frac{1}{R_1 C_1}} \qquad \text{(+1 point)}$$

and substitute $C_1 = 1/L_1$:

$$\begin{aligned} T(1/s) &= -\frac{L_1}{R_2} \times \frac{1}{\frac{1}{s} + \frac{L_1}{R_1}} \\ &= -\frac{R_1}{R_2} \times \frac{s}{s + \frac{R_1}{L_1}} = \tilde{T}(s) \end{aligned} \qquad \text{(+1 point)}$$

after some calculations.

(c) (1 point (bonus)) What happens to the poles and zeros of the transfer function $T(s)$ as it gets transformed to $\tilde{T}(s) = T(1/s)$? What happens to the frequency response?

Solution:

Because when $C_1 = 1/L_1$

$$\frac{R_1}{L_1} = \frac{1}{\frac{1}{R_1 C_1}}$$

the location of the original pole is inverted after flipping the circuit. (+0.5 point)

The transfer function $\tilde{T}(s) = T(1/s)$ has a zero at $s = 0$ while $T(s)$ has no zeros. (+0.5 point)

Some people like to think of $T(s)$ as having originally a "zero at infinity" which is now "inverted" to produce a zero at $s = 0$.

(d) (4 points) What happens to the transfer-function of the circuit in Fig. 2a if you replace the capacitor C_1 by a capacitor $\alpha\, C_1$, where $\alpha > 0$? Verify that the new transfer-function is

$$\tilde{T}(s) = T(\alpha\, s).$$

Solution:

Replace C_1 by αC_1 in the circuit, convert to the s-domain, etc (+2 points)

The obtain the transfer function

$$\begin{aligned} \tilde{T}(s) &= -\frac{1}{\alpha R_2 C_1} \times \frac{1}{s + \frac{1}{\alpha R_1 C_1}} && \text{(+1 point)} \\ &= -\frac{1}{R_2 C_1} \times \frac{1}{\alpha s + \frac{1}{R_1 C_1}} = T(\alpha s) && \text{(+1 point)} \end{aligned}$$

after multiplication by α on the numerator and denominator.

(e) (1 point (bonus)) What happens to the poles and zeros of the transfer function $T(s)$ as it gets transformed to $\tilde{T}(s) = T(\alpha\, s)$? What happens to the frequency response?

Solution:

Because

$$\frac{1}{\alpha R_1 C_1} = \frac{1}{\alpha} \times \frac{1}{R_1 C_1}$$

the original location of the pole is divided by α. (+0.5 point)

No zeros are created by the transformation $\tilde{T}(s) = T(\alpha s)$. (+0.5 point)

But any existing zeros would be scaled as the poles did.