Solutions HW 6

9.1

$$f(t) = 500[1 - e^{-100t}]u(t)$$

$$= 500\,u(t) - 500\,e^{-100t}\,u(t)$$

(One can say that there is a double zero at infinity because asymptotically $F(s)$ is proportional to $\frac{1}{s^2}$)

Using linearity, we have

$$F(s) = 500\,\mathcal{L}\{u(t)\} - 500\,\mathcal{L}\{e^{-100t}u(t)\}$$

you can use a table

$$\overset{\downarrow}{=} 500\,\frac{1}{s} - 500\,\frac{1}{s+100} = \frac{500 \cdot 100}{s(s+100)}$$

$\boxed{\text{zeros: } \infty \text{ double} \quad \text{poles: } 0, -100}$

9.4

$$f(t) = 20\,[e^{-200t} - 2e^{-100t}]\,u(t)$$

$$F(s) = 20\left(\mathcal{L}\{e^{-200t}u(t)\} - 2\,\mathcal{L}\{e^{-100t}u(t)\}\right)$$

$$= 20\left(\frac{1}{s+200} - 2\,\frac{1}{s+100}\right) = -\frac{20(s+300)}{(s+200)(s+100)}$$

* because $F(s)$ is proportional to $\frac{1}{s}$ when $s \to \infty$

$\boxed{\text{Zeros: } \infty^*, -300 \text{ ; poles} = -200, -100}$

9.6

$$f(t) = 0.005\,[10 - 10\cos(1000t)]\,u(t)$$

$$F(s) = 0.005\left(10\,\mathcal{L}\{u(t)\} - 10\,\mathcal{L}\{\cos(1000t)\,u(t)\}\right)$$

$$= 0.05\left(\frac{1}{s} - \frac{s}{s^2 + (10^3)^2}\right) = 0.05\,\frac{10^6}{s(s^2+10^6)}$$

$\boxed{\text{Zeros}^*: \infty \text{ (triple) ; poles: } 0, 10^3 j, -10^3 j}$

* (three zeros at ∞ because asymptotically $F(s)$ behaves like $\frac{1}{s^3}$)

$\boxed{9.11}$ a) $f_1(t) = 2\delta(t) + \left[200e^{-200t} + 400e^{-400t}\right]u(t)$

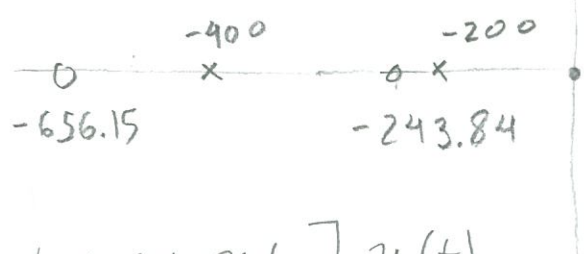
$$F_1(s) = 2\mathcal{L}\{\delta(t)\} + 200\mathcal{L}\{e^{-200t}u(t)\} + 400\mathcal{L}\{e^{-400t}u(t)\}$$

$$= 2 + \frac{200}{s+200} + \frac{400}{s+400} = \frac{2(s^2+600s+8\times10^4) + 600s + 16\times10^4}{(s+200)(s+400)}$$

$$= \frac{2s^2 + 1800s + 32\times10^4}{(s+200)(s+400)}$$

$\boxed{\text{zeros: } -656.15,\ -243.84 \quad \text{poles: } -400,\ -200}$

Pole zero map:



this was 9.10 b)

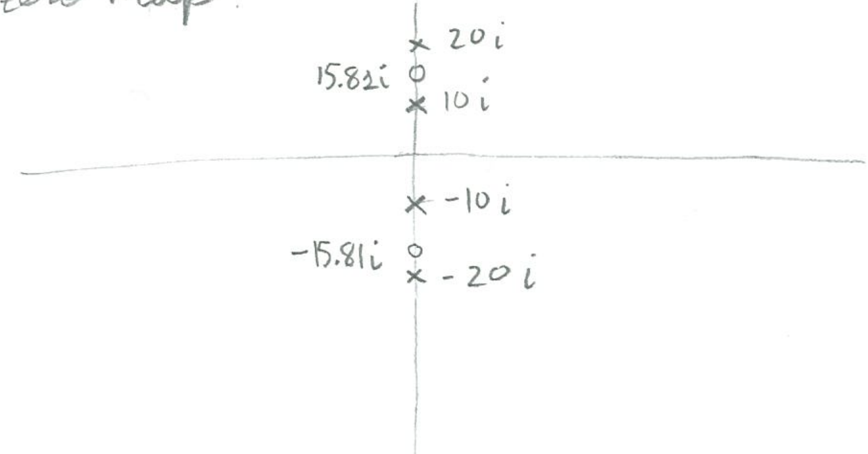
b) $f_2(t) = 10[\cos 10t + \cos 20t]\,u(t)$

$$F_2(s) = 10\mathcal{L}\{\cos(10t)\,u(t)\} + 10\mathcal{L}\{\cos(20t)\,u(t)\}$$

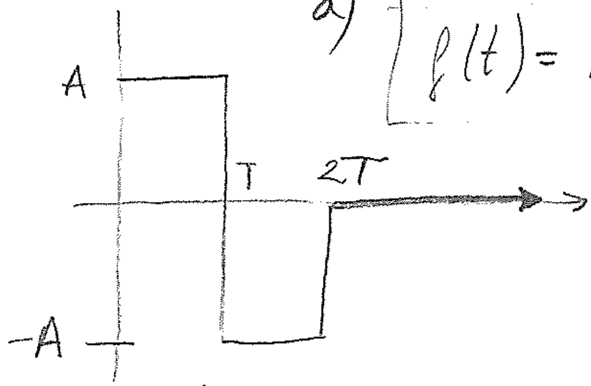
$$= 10\frac{s}{s^2+10^2} + 10\frac{s}{s^2+20^2} = \frac{10s(2s^2+500)}{(s+10i)(s-10i)(s+20i)(s-20i)}$$

$\boxed{\text{zeros: } 0,\ \pm\sqrt{\tfrac{500}{2}}\,i\ ;\quad \text{poles: } \pm10i,\ \pm20i}$

Pole-zero map:



9.17



a) $\boxed{f(t) = A\cdot\left(u(t) - 2u(t-T) + u(t-2T)\right)}$

b)
$$F(s) = A\left(\mathcal{L}\{u(t)\} - 2\mathcal{L}\{u(t-T)\} + \mathcal{L}\{u(t-2T)\}\right)$$
$$= A\left(\frac{1}{s} - \frac{2e^{-Ts}}{s} + \frac{e^{-2Ts}}{s}\right)$$
$$= \boxed{\frac{A}{s}\left(1 - 2e^{-Ts} + e^{-2Ts}\right)}$$

c) To check that $F(s) = \mathcal{L}\{f(t)\}$ using the definition $F(s) = \int_{0^-}^{\infty} f(t) e^{-ts}\, dt$, we can rely on the linearity property. Hence, we just need to check using the definition that

$$\boxed{\mathcal{L}\{u(t-a)\}} = \int_{0^-}^{\infty} u(t-a) e^{-ts}\, ds = \int_{a}^{\infty} u(t-a) e^{-ts}\, ds$$

$$\overset{\tau = t-a}{=} \int_{\tau=0}^{\tau=\infty} u(\tau) e^{-(\tau+a)s}\, ds = e^{-as}\mathcal{L}\{u(\tau)\}$$
$$= \boxed{\frac{e^{-as}}{s}},$$

as needed.

$\boxed{9.20}$ a) $F_1(s) = \dfrac{50}{s(s+50)}$

It is proper and has different poles, so we can write

$$F_1(s) = \frac{K_1}{s} + \frac{K_2}{s+50} \quad \text{where}$$

$$K_1 = \lim_{s \to 0} F_1(s) \cdot s = \lim_{s \to 0} \frac{50}{(s+50)} = 1$$

$$K_2 = \lim_{s \to -50} F_1(s)(s+50) = \lim_{s \to -50} \frac{50}{s} = -1$$

I.e., $F_1(s) = \dfrac{1}{s} - \dfrac{1}{s+50}$, which yields

$$\mathcal{L}^{-1}\{F_1(s)\} = \boxed{u(t) - e^{-50t}u(t)}$$

b) $F_2(s) = \dfrac{s+1}{(s+2)(s+3)}$

Again, it is proper and with different poles, so we write

$$F_2(s) = \frac{K_1}{s+2} + \frac{K_2}{s+3} \quad \text{where } K_1 = \frac{-2+1}{-2+3} = -1$$

and $K_2 = \dfrac{-3+1}{-3+2} = \dfrac{-2}{-1} = 2$ so

$$\mathcal{L}^{-1}\{F_2(s)\} = \boxed{\left(-e^{-2t} + 2e^{-3t}\right)u(t)}$$

9.22 a) $F_1(s) = \dfrac{5000(s+10^3)}{(s+500)(s+5\times10^3)}$

Since it is proper and w. different poles, it can be expressed as

$$F_1(s) = \frac{K_1}{s+500} + \frac{K_2}{s+5\times10^3},$$

where $K_1 = \lim_{s\to-500} F(s)\cdot(s+500) = \dfrac{5\times10^3(-5\times10^2+10^3)}{(-5\times10^2+5\times10^3)} = 555.5$

$K_2 = \lim_{s\to-5\times10^3} F(s)(s+5\times10^3) = \dfrac{5\times10^3(-5\times10^3+10^3)}{-5\times10^3+500}$

$= 4.4\times10^3$

That is, $F_1(s) = \dfrac{555.5}{s+500} + \dfrac{4.4\times10^3}{s+5000}$

so $\boxed{f_1(t) = \mathcal{L}^{-1}\{F_1(s)\} = 555.5\, e^{-500t} + 4.4\times10^3\, e^{-5000t}},$

b) $F_2(s) = \dfrac{5s^2}{(s+100)(s+500)}$

Again, it has different poles but is not proper, so

we divide $5s^2 \,\big|\, s^2+600s+5\times10^4$

$$\begin{array}{l} 5s^2 \\ -\ 5s^2 + 3\times10^3 s + 2.5\times10^5 \quad 5 \\ \hline -3\times10^3 s - 2.5\times10^5 \end{array}$$

$\hat{F}_2(s)$

so $F_2(s) = 5 - \dfrac{3\times10^3 s + 2.5\times10^5}{(s+100)(s+500)} = 5 - \left(\dfrac{K_1}{s+100} + \dfrac{K_2}{s+500}\right)$

$\boxed{K_1} = \lim_{s\to-100} \hat{F}_2(s)(s+100) = \dfrac{3\times10^3(-100)+2.5\times10^5}{-100+500} = \dfrac{(-3+2.5)\times10^5}{400} = \boxed{=-125}$

$\boxed{K_2} = \lim_{s\to-500} \hat{F}_2(s)(s+500) = \dfrac{3\times10^3(-500)+2.5\times10^5}{-500+100} = \dfrac{(-15+2.5)10^5}{-400} = \boxed{=3125}$

So $\boxed{f_2(t) = 5\delta(t) + 125\, e^{-100t} - 3125\, e^{-500t}}$

$\boxed{9.24}$ a) $F_1(s) = \dfrac{\beta(s+\beta)}{s(s^2+\beta^2)}$

It is proper and has different poles

$$F_1(s) = \frac{K_1}{s} + \frac{K_2}{s+\beta i} + \frac{K_3}{s-\beta i}$$

$$K_1 = \lim_{s\to 0} \cancel{s}\frac{\beta(s+\beta)}{\cancel{s}(s^2+\beta^2)} = \frac{\beta^2}{\beta^2} = 1$$

$$K_2 = \lim_{s\to -\beta i} \cancel{(s+\beta i)}\frac{\beta(s+\beta)}{s\cancel{(s+\beta i)}(s-\beta i)} = \frac{\beta(-\beta i+\beta)}{(-\beta i)(-2\beta i)}$$

$$= \frac{-\cancel{\beta^2} i + \cancel{\beta^2}}{-2\cancel{\beta^2}} = -\left(\frac{-i+1}{2}\right) = -\frac{1}{2} + \frac{i}{2}$$

$$K_3 = \overline{K_2} \text{ (conjugate)} = \overline{\left(-\frac{1}{2}-\frac{i}{2}\right)} = -\frac{1}{2} - \frac{i}{2}$$

Hence $f_1(t) = u(t) + \left(-\frac{1}{2}+\frac{i}{2}\right)e^{-\beta i t}u(t) + \left(-\frac{1}{2}-\frac{i}{2}\right)e^{\beta i t}u(t)$

$$= u(t)\left[1 - \frac{1}{2}\left(e^{-\beta i t} + e^{\beta i t}\right) + \frac{i}{2}e^{-\beta i t} + \frac{-i}{2}e^{\beta i t}\right]$$

$$= u(t)\left[1 - \cos\beta t + \frac{-e^{-\beta i t} + e^{\beta i t}}{2i}\right]$$

multiplying by $\frac{i}{i}$

$$= u(t)\left[1 - \cos\beta t + \sin\beta t\right]$$

Note: There are equivalent expressions like $-\sqrt{2}\sin\left(\frac{\pi}{4}-\beta t\right)$,
One can get these expressions taking the polar form of the residues.

$\boxed{9.24}$ b) $F_2(s) = \dfrac{s(s+\beta)}{s^2+\beta^2}$

It has <u>different poles</u> but is <u>not proper</u>

so we need to divide

$$\begin{array}{l|l} s^2 + s\beta & s^2+\beta^2 \\ \hline -\; s^2 + \beta^2 & 1 \\ \hline \cancel{}\; s\beta - \beta^2 & \end{array}$$

to obtain $F_2(s) = 1 + \underbrace{\dfrac{\beta(s-\beta)}{s^2+\beta^2}}_{=\hat{F}_2(s)} = 1 + \dfrac{K_1}{s+\beta i} + \dfrac{K_2}{s-\beta i}$

where

$$K_1 = \lim_{s\to -\beta i} \hat{F}_2(s)(s+\beta i) = \lim_{s\to -i\beta} \frac{\beta(s-\beta)}{(s-i\beta)} = \frac{\cancel{\beta}(-i\beta-\beta)}{-2i\cancel{\beta}}$$

$$= \frac{i}{i}\,\frac{-i\beta-\beta}{-2i} = \frac{\beta-\beta i}{2}$$

$$K_2 = \overline{K_1} = \frac{\beta+\beta i}{2}$$

Hence $F_2(s) = 1 + \dfrac{\frac{\beta}{2} - \frac{\beta}{2} i}{s+i\beta} + \dfrac{\frac{\beta}{2} + \frac{\beta}{2} i}{s-i\beta}$, that

yields

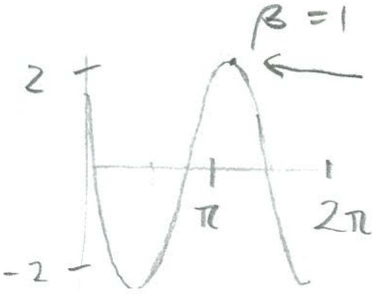
$$f_2(t) = \mathcal{L}^{-1}\{F_2(s)\} = \delta(t) + \left(\tfrac{\beta}{2} - \tfrac{\beta}{2} i\right)e^{-i\beta t} + \left(\tfrac{\beta}{2}+\tfrac{\beta}{2} i\right)e^{i\beta t}$$

$$= \delta(t) + \beta\left(\frac{e^{-i\beta t}+e^{i\beta t}}{2}\right) + \beta i\left(\frac{-e^{-i\beta t}+e^{i\beta t}}{2}\right)$$

$$= \boxed{\delta(t) + \beta\cos(\beta t) - \beta\sin\beta t}$$

multiplying by $\frac{i}{i}$

<u>Matlab plot without $\delta(t)$</u>



The max is not at π, there is a phase.

There are alternate forms:

$$\delta(t) + \beta\sqrt{2}\sin\left(\frac{\pi}{4} - \beta t\right)$$

$\boxed{9.25}$ a) $F_1(s) = \dfrac{\alpha^2}{s^2(s+\alpha)}$

It is proper and 0 is a repeated pole

$$F_1(s) = \frac{K_1}{s} + \frac{K_2}{s^2} + \frac{K_3}{s+\alpha}$$

$$K_1 = \lim_{s \to 0} \frac{d}{ds}\left[s^2 \frac{\alpha^2}{\cancel{s^2}(s+\alpha)} \right] = \lim_{s \to 0} -\frac{\alpha^2}{(s+\alpha)^2} = -1$$

$$K_2 = \lim_{s \to 0} \cancel{s^2} \frac{\alpha^2}{\cancel{s^2}(s+\alpha)} = \alpha$$

$$K_3 = \lim_{s \to -\alpha} \cancel{(s+\alpha)} \frac{\alpha^2}{s^2\cancel{(s+\alpha)}} = 1.$$

So $F_1(s) = \dfrac{-1}{s} + \dfrac{\alpha}{s^2} + \dfrac{1}{s+\alpha}$.

Hence $f_1(t) = \mathcal{L}^{-1}\{F_1(s)\} =$

$$= -u(t) + \alpha t\, u(t) + e^{-\alpha t} u(t)$$

$$= u(t) \left[-1 + \alpha t + e^{-\alpha t} \right]$$

19.25 b) $F_2(s) = \dfrac{\alpha^2}{s(s+\alpha)^2} = \dfrac{K_1}{s} + \dfrac{K_2}{s+\alpha} + \dfrac{K_3}{(s+\alpha)^2}$

(Note that F_2 is proper but $-\alpha$ is a repeated pole)

$$K_1 = \lim_{s \to 0} F_2(s)\, s = \frac{\alpha^2}{\alpha^2} = 1$$

$$K_2 = \lim_{s \to -\alpha} \frac{d}{ds} \cancel{(s+\alpha)^2} \frac{\alpha^2}{s\cancel{(s+\alpha)^2}} = \lim_{s \to -\alpha} -\frac{\alpha^2}{s^2} = -1$$

$$K_3 = \lim_{s \to -\alpha} \cancel{(s+\alpha)^2} \frac{\alpha^2}{s\cancel{(s+\alpha)^2}} = \frac{\alpha^2}{-\alpha} = -\alpha .$$

I.e., $F_2(s) = \dfrac{1}{s} - \dfrac{1}{s+\alpha} - \dfrac{\alpha}{(s+\alpha)^2}$,

which yields $f_2(t) = \mathcal{L}^{-1}\{F_2(s)\} =$

$$= u(t) - e^{-\alpha t} u(t) - \alpha t e^{-\alpha t} u(t)$$

$$= u(t) \left[1 - e^{-\alpha t} - \alpha t e^{-\alpha t} \right]$$

$\boxed{9.34}$ a) $F_1(s) = \dfrac{s^2}{s+5}$

which is not proper, so we compute

$$\begin{array}{r|l} s^2 & s+5 \\ \hline -\ (\cancel{s^2}+5s) & s-5 \\ \hline -5s & \\ -\ (-5s-25) & \\ \hline 25 & \end{array}$$

that is, $F_1(s) = s-5+\dfrac{25}{s+5}$, so

$$f_1(t) = \mathcal{L}^{-1}\{F_1(s)\} = \boxed{\frac{d\delta(t)}{dt} - 5\delta(t) + 25e^{-5t}}$$

b) $F_2(s) = \dfrac{(s+10^3)^2}{(s+2\times10^3)^2}$, which is not proper.

$$\begin{array}{r|l} s^2+2000s+10^6 & s^2+4000s+4\times10^6 \\ \hline -\ (s^2+4000s+4\times10^6) & 1 \\ \hline -2000s-3\times10^6 & \end{array}$$

so $F_2(s) = 1 - \dfrac{2000s+3\times10^6}{(s+2000)^2} = \dfrac{K_1}{(s+2000)} + \dfrac{K_2}{(s+2000)^2}$

$$K_1 = \lim_{s\to-2000}\frac{d}{ds}\left[2000s+3\times10^6\right] = 2000$$

$$K_2 = \lim_{s\to-2000} 2000s+3\times10^6 = -4\times10^6+3\times10^6 = -10$$

That is, $F_2(s) = 1 - \dfrac{2000}{s+2000} + \dfrac{10^6}{(s+2000)^2}$

Therefore, $f_2(t) = \boxed{\delta(t) - 2000e^{-2000t} + 10^6te^{-2000t}}$