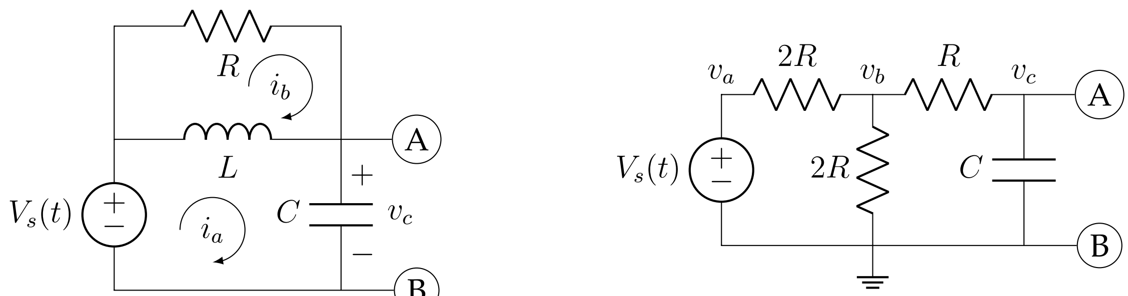
MAE140 – Linear Circuits – Fall – 2019
Final, December 12th

Instructions

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a handheld calculator with no communication capabilities.
2. You have 170 minutes.
3. Do not forget to write your name and student number.
4. This exam has 6 questions, for a total of 60 points.



(a) (b)

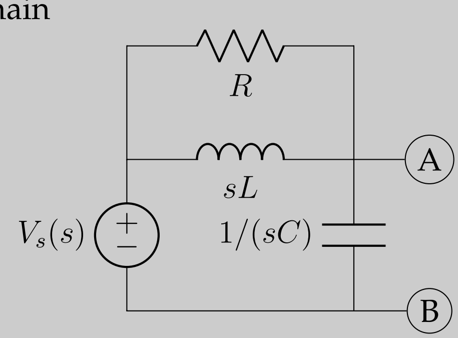
Figure 1: Circuits for Questions 1-3

1. **Circuit Equivalence - Part I**

   Consider the circuit in Fig. 1a and answer the following questions:

   (a) (1 point) Assume zero initial conditions and convert the circuit into the s-domain.

   Solution: In the s-domain



   (+1 point)

   (b) (3 points) Calculate the s-domain open-circuit voltage at (A) and (B).

Solution: For the open-circuit voltage calculation the circuit is as in part (a). (+1 point)

One can use voltage division after associating R and sL in parallel

$$Z(s) = \frac{1}{\frac{1}{R} + \frac{1}{sL}} = \frac{sLR}{R + sL} \qquad \text{(+1 point)}$$

it follows that

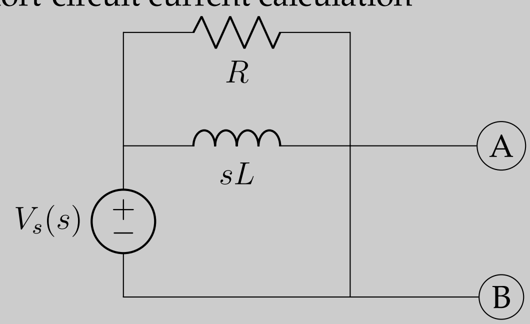
$$V_{AB}(s) = \frac{\frac{1}{sC}}{\frac{1}{sC} + Z(s)} V_s(s) = \frac{\frac{1}{sC}}{\frac{1}{sC} + \frac{sLR}{R+sL}} V_s(s) = \frac{R + sL}{s^2 LRC + sL + R} V_s(s)$$

(+1 point)

NOTE TO GRADERS: Students get full marks for the correct reasoning no matter how simplified their equations look like.

(c) (3 points) Calculate the s-domain short-circuit current at A and B.

Solution: For the short-circuit current calculation



(+1 point)

From which the short-circuit current is the sum of the currents on R and sL. After associating

$$Z(s) = \frac{1}{\frac{1}{R} + \frac{1}{sL}} = \frac{sLR}{R + sL} \qquad \text{(+1 point)}$$

it follows that

$$I_{AB}(s) = \frac{V_s(s)}{Z(s)} = \frac{R + sL}{sLR} V_s(s) \qquad \text{(+1 point)}$$

(d) (2 points) Calculate the equivalent impedance as seen from A and B.

Solution: The equivalent impedance is obtaied by dividing the open-circuit voltage by the short-circuit current. (+1 point)

That is

$$Z_{eq}(s) = \frac{V_{AB}(s)}{I_{AB}(s)} = \frac{R + sL}{s^2LRC + sL + R}\frac{sLR}{R + sL} = \frac{sLR}{s^2LRC + sL + R} \qquad \text{(+1 point)}$$

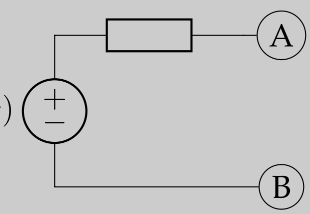
NOTE TO GRADERS: Students get full marks for the correct reasoning no matter how simplified their equations look like.

(e) (1 point) Draw the s-domain Thevenin equivalent for the circuit in Fig. 1a.

Solution: The Thevenin equivalent circuit should look like

$Z_{eq}(s) = \frac{sLR}{s^2LRC+sL+R}$

$V_T(s) = V_{AB}(s) = \frac{R+sL}{s^2LRC+sL+R}V_s(s)$



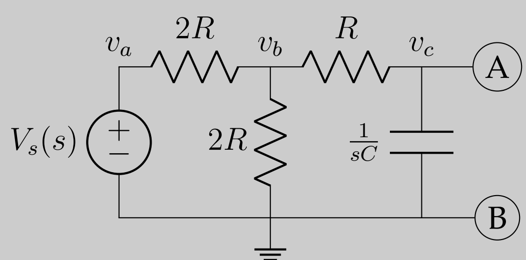
(+1 point)

2. **Circuit Equivalence - Part II**

Consider the circuit in Fig. 1b and answer the following questions:

(a) (1 point) Assume zero initial conditions and convert the circuit into the s-domain.

Solution: In the s-domain



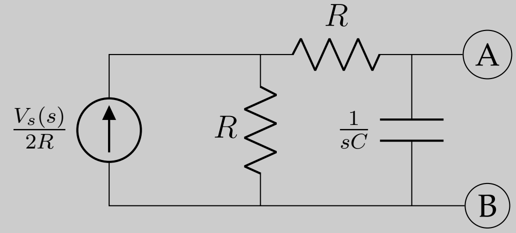
(+1 point)

(b) (3 points) Calculate the s-domain open-circuit voltage at A and B.

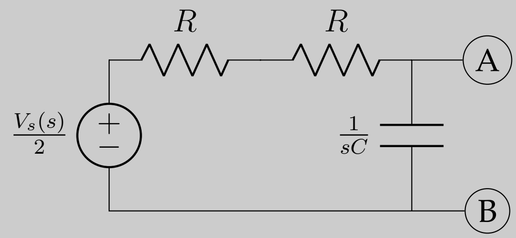
Solution: For the open-circuit voltage calculation the circuit is as in part (a).

(+1 point)

After a series of equivalences



(+1/2 point)



(+1/2 point)

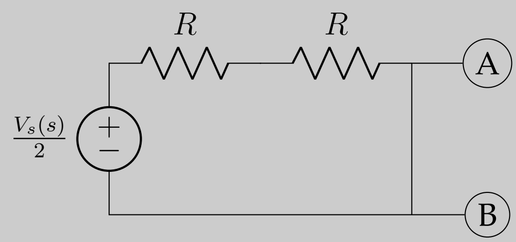
From voltage division

$$V_{AB}(s) = \frac{\frac{1}{sC}}{\frac{1}{sC} + 2R}\frac{V_s(s)}{2} = \frac{\frac{1}{4RC}}{s + \frac{1}{2RC}}V_s(s) \qquad \text{(+1 point)}$$

NOTE TO GRADERS: Students get full marks for the correct reasoning no matter how simplified their equations look like.

(c) (3 points) Calculate the s-domain short-circuit current at A and B.

Solution: For the short-circuit current calculation we can start from the last circuit in part (b):



(+1 point)

From which $I_{AB}(s)$ is the current on the resistors. (+1 point)

That is

$$I_{AB}(s) = \frac{V_s(s)/2}{2R} = \frac{V_s(s)}{4R} \qquad \text{(+1 point)}$$

(d) (2 points) Calculate the equivalent impedance as seen from A and B.

Solution: The equivalent impedance is obtaied by dividing the open-circuit voltage by the short-circuit current. (+1 point)

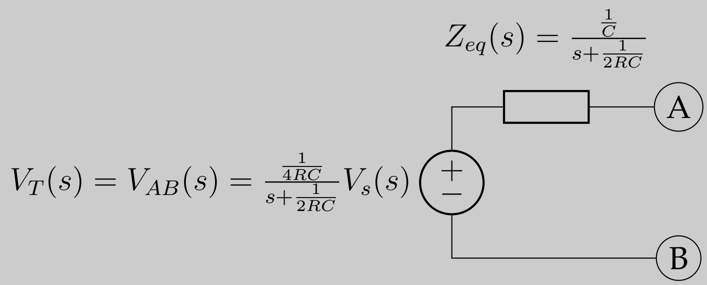
That is

$$Z_{eq}(s) = \frac{V_{AB}(s)}{I_{AB}(s)} = 4R\frac{\frac{1}{4RC}}{s + \frac{1}{2RC}} = \frac{\frac{1}{C}}{s + \frac{1}{2RC}} \qquad \text{(+1 point)}$$

NOTE TO GRADERS: Students get full marks for the correct reasoning no matter how simplified their equations look like.

(e) (1 point) Draw the s-domain Thevenin equivalent for the circuit in Fig. 1b.

Solution: The Thevenin equivalent circuit should look like

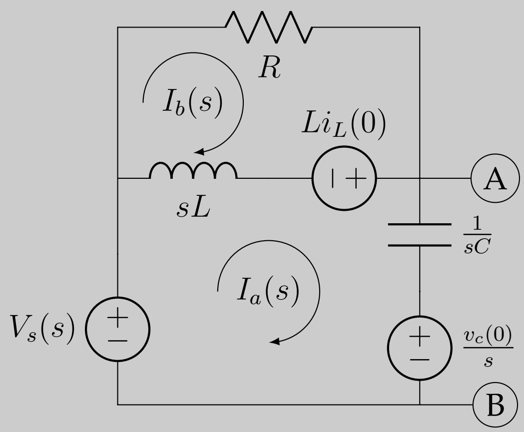


(+1 point)

3. **Circuit Analysis**

(a) (4 points) Do not assume zero-initial conditions, convert the circuit in Fig. 1a to the s-domain and formulate its mesh-current equations. Express the component's initial conditions in terms of the indicated mesh-currents and voltage. Use the mesh-currents and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns. Make sure your final equations only involve mesh-currents.

Solution:
In the s-domain



in which

$$i_L(0) = i_a(0) - i_b(0). \qquad \text{(+1 point)}$$

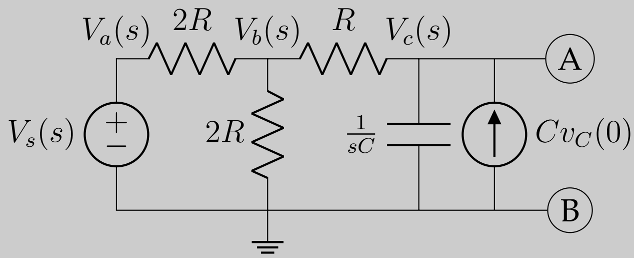
Writing KVL by inspection

$$\begin{bmatrix} sL + \frac{1}{sC} & -sL \\ -sL & R + sL \end{bmatrix} \begin{pmatrix} I_a(s) \\ I_b(s) \end{pmatrix} = \begin{pmatrix} V_s(s) + L(i_a(0) - i_b(0)) - \frac{v_c(0)}{s} \\ -L(i_a(0) - i_b(0)) \end{pmatrix} \qquad \text{(+2 points)}$$

which are the node-voltage equations in the circuit unknowns $I_a(s)$ and $I_b(s)$.
(+1 point)

(b) (5 points) Do not assume zero-initial conditions, convert the circuit in Fig. 1b to the s-domain and formulate its node-voltage equations. Express the component's initial conditions in terms of the indicated node-voltages. Use the node-voltages and labels provided in the figure and clearly indicate the final equations and circuit variable unknowns. Make sure your final equations only involve node-voltages.

Solution:
In the s-domain:



(+1 point)

We write the node-voltage equations by inspection omitting node a:

$$\begin{bmatrix} -\frac{1}{2R} & \frac{1}{2R}+\frac{1}{2R}+\frac{1}{R} & -\frac{1}{R} \\ 0 & -\frac{1}{R} & \frac{1}{R}+sC \end{bmatrix} \begin{pmatrix} V_a(s) \\ V_b(s) \\ V_c(s) \end{pmatrix} = \begin{pmatrix} 0 \\ Cv_c(0) \end{pmatrix} \qquad \text{(+1 point)}$$

along with the additional relation:

$$V_a(s) = V_s(s) \qquad \text{(+1 point)}$$

The above equations need to be solved for the node voltages $V_a(s)$, $V_b(s)$ and $V_c(s)$. (+1 point)

(c) (1 point) Write s-domain expressions for the voltage A and B in terms of the node voltages and mesh currents from parts (a) and (b).

Solution:
In circuit (a):

$$V_{AB}(s) = \frac{1}{sC} I_a(s) + \frac{v_c(0)}{s} \qquad \text{(+1/2 point)}$$

In circuit (b):

$$V_{AB}(s) = V_c(s) \qquad \text{(+1/2 point)}$$

4. **Frequency Response**

Consider the transfer-function:

$$T(s) = \frac{10s^2}{s^2 + s\,200 + 10{,}000}$$

and answer the following questions:

(a) (2 points) What are the poles and zeros of $T(s)$?

Solution: The poles are the roots of the equation

$$0 = (s^2 + s200 + 10000) = (s+100)(s+100)$$

that is $\{-100, -100\}$. (+1 point)

The zeros are the roots of the equation

$$0 = s^2$$

that is $\{0, 0\}$. (+1 point)

(b) (1 point) Is $T(s)$ asymptotically stable?

Solution: Both poles, $\{-100, -100\}$, have negative real part, therefore $T(s)$ is asymptotically stable. (+1 point)

(c) (4 points) Evaluate $|T(j\omega)|$ and $\angle T(j\omega)$ at $\omega = \{0, 100, \infty\}$.

Solution: For $\omega \geq 0$

$$T(j\omega) = \frac{-10\,\omega^2}{(j\omega + 100)^2}, \quad |T(j\omega)| = \frac{10\,\omega^2}{\omega^2 + 100^2}, \quad \angle T(j\omega) = \pi - 2\tan^{-1}\omega/100,$$

(+1 point)

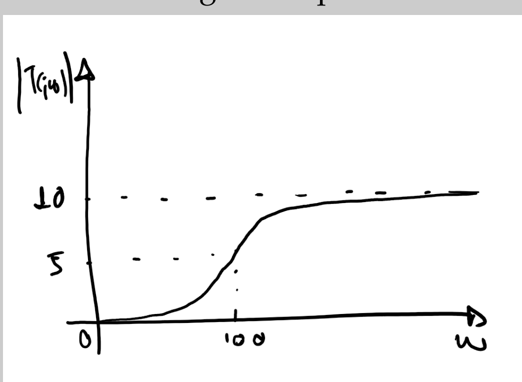
Evaluating

ω	0	100	∞
$\lvert T(j\omega)\rvert$	0	$\frac{10}{2}$	10
$\angle T(j\omega)$	π	$\pi/2$	0

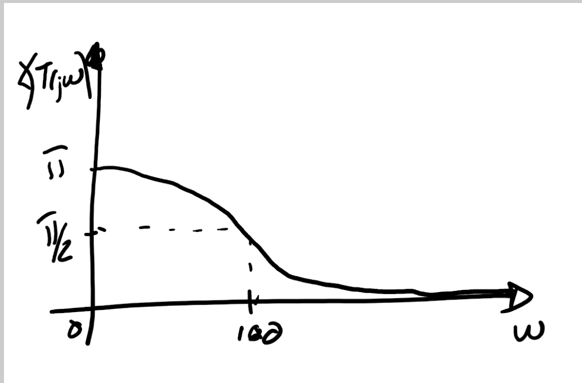
(+3 points)

(d) (2 points) Sketch the functions $|T(j\omega)|$ and $\angle T(j\omega)$ for $\omega \geq 0$.

Solution: Magnitude plot should look like



Phase plot should look like



(e) (1 point) What type of filter is $T(s)$?

Solution: These are high-pass filters. (+1 point)

5. (10 points) **OpAmp Design**

Consider the transfer-function:

$$T(s) = \frac{10s^2}{s^2 + s\,200 + 10{,}000}$$

from Question 4 and design a circuit using opAmps that implements $T(s)$.

Solution: One simple solution is to look at the transfer-function as the cascade circuits

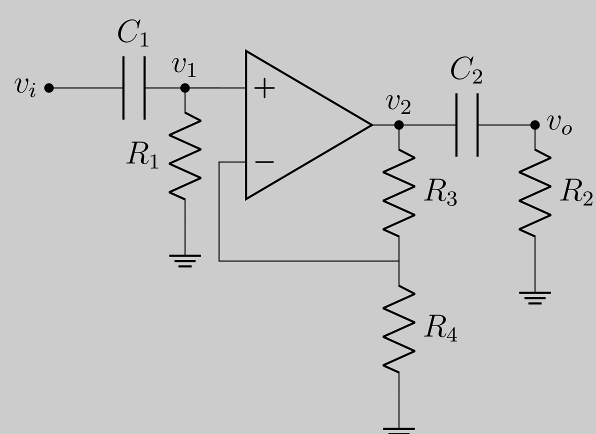
$$T(s) = \frac{10s^2}{s^2 + s200 + 10000} = \frac{10s^2}{(s+100)^2} = 10\frac{s}{(s+100)}\frac{s}{(s+100)}$$

with two first-order high-pass filters

$$T_1(s) = T_2(s) = \frac{s}{s+100}$$

and an additional gain of 10.

A possible circuit is the following



In this circuit

$$V_1(s) = \frac{s}{s + \frac{1}{R_1 C_1}} V_i(s), \qquad V_2(s) = \frac{R_3 + R_4}{R_4} V_1(s), \qquad V_o(s) = \frac{s}{s + \frac{1}{R_2 C_2}} V_2(s),$$

so that

$$V_o(s) = \frac{R_3 + R_4}{R_4} \frac{s}{s + \frac{1}{R_2 C_2}} \frac{s}{s + \frac{1}{R_1 C_1}} V_i(s)$$

which solves the problem after selecting R_1, R_2, R_3, R_4, C_1, and C_2 as in

$$\frac{1}{R_1 C_1} = \frac{1}{R_2 C_2} = 100, \qquad \frac{R_3 + R_4}{R_4} = 10.$$

(+10 points)

NOTE TO GRADERS: Multiple solutions are possible. Students have to explain why their circuit solves the problem and how to calculate component values.

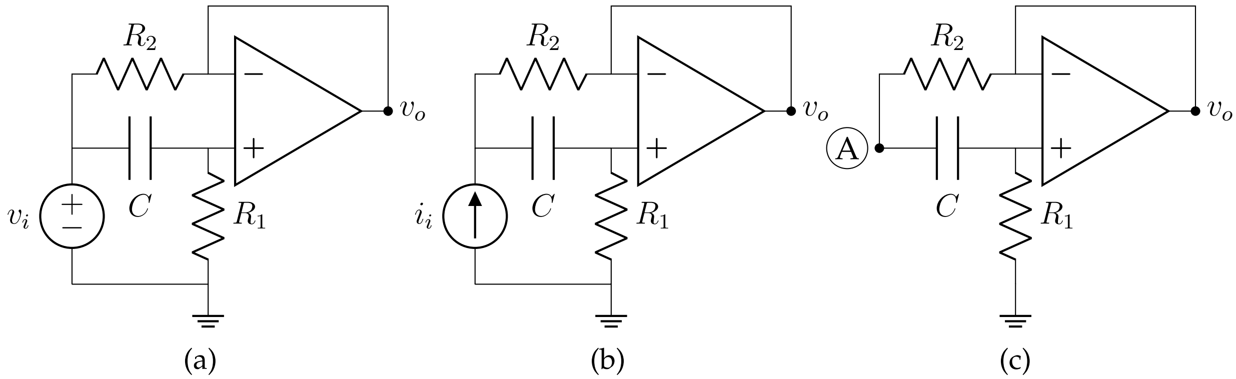


Figure 2: Circuits for Question 6

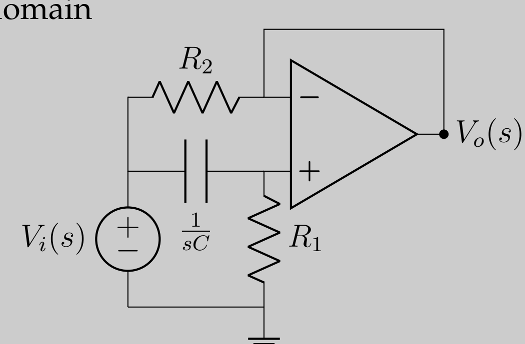
6. **OpAmp Analysis**

   Assume zero initial-conditions for the circuits in Fig. 2 and answer the questions:

   (a) (3 points) For the circuit in Fig. 2a show that

$$\frac{V_o(s)}{V_i(s)} = \frac{s}{s + \frac{1}{R_1 C}}.$$

Solution: In the s-domain



(+1 point)

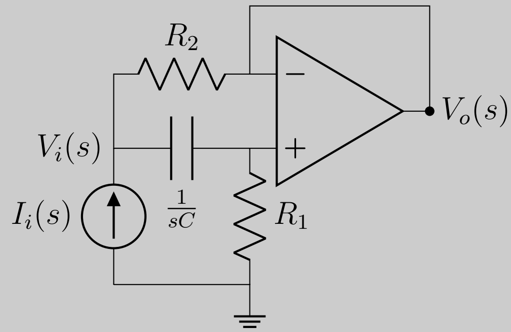
The expression in the statement follows from

$$V_0(s) = V_-(s) = V_+(s) = \frac{R_1}{R_1 + \frac{1}{sC}} V_i(s) = \frac{s}{s + \frac{1}{R_1 C}} V_i(s)$$ (+2 points)

   (b) (3 points) For the circuit in Fig. 2b show that

$$\frac{V_o(s)}{I_i(s)} = \frac{sR_1}{s + \frac{1}{R_2 C}}.$$

Solution: In the s-domain



(+1 point)

The expression in the statement follows from

$$V_o(s) = V_-(s) = V_+(s) = \frac{R_1}{R_1 + \frac{1}{sC}} V_i(s) = \frac{s}{s + \frac{1}{R_1 C}} V_i(s) \qquad \text{(+1/2 point)}$$

and

$$I_1(s) = \frac{V_i(s) - V_0(s)}{R_2} + sC(V_i(s) - V_0(s)) = \left(\frac{1}{R_2} + sC\right)(V_i(s) - V_0(s))$$

(+1/2 point)

Combining the above

$$\frac{1}{\frac{1}{R_2} + sC} I_1(s) = V_i(s) - V_0(s) = \left(\frac{s + \frac{1}{R_1 C}}{s} - 1\right) V_o(s) = \frac{1}{sR_1 C} V_o(s)$$

(+1/2 point)

or

$$V_o(s) = \frac{sR_1}{s + \frac{1}{R_2 C}} I_1(s) \qquad \text{(+1/2 point)}$$

(c) (2 points) Show that the impedance at point A in Fig. 2c is

$$Z(s) = R_1 \frac{s + \frac{1}{R_1 C}}{s + \frac{1}{R_2 C}}.$$

Solution: The impedance as seen from A (and ground) is going to be the ratio

$$Z(s) = \frac{V_i(s)}{I_s(s)} \qquad \text{(+1 point)}$$

which can be readly calculated from above

$$Z(s) = \frac{V_o(s)/I_i(s)}{V_o(s)/V_i(s)} = \frac{sR_1}{s + \frac{1}{R_2 C}} \frac{s + \frac{1}{R_1 C}}{s} = R_1 \frac{s + \frac{1}{R_1 C}}{s + \frac{1}{R_2 C}} \qquad \text{(+1 point)}$$

(d) (1 point) Show that

$$Z(j\omega) \approx R_2 + R_1 R_2 C j\omega$$

when $\omega \ll \frac{1}{R_2 C}$.

Hint: What is the value of

$$\frac{\frac{1}{R_2 C}}{j\omega + \frac{1}{R_2 C}}$$

when $\omega \ll \frac{1}{R_2 C}$?

Solution: Because

$$\frac{\frac{1}{R_2 C}}{j\omega + \frac{1}{R_2 C}} \approx 1$$

when $\omega \ll \frac{1}{R_2 C}$

$$Z(j\omega) = R_1 R_2 C \left(j\omega + \frac{1}{R_1 C} \right) \frac{\frac{1}{R_2 C}}{j\omega + \frac{1}{R_2 C}} \approx R_1 R_2 C j\omega + R_2 \qquad \text{(+1 point)}$$

(e) (1 point) Explain why the above circuit is called a "simulated inductor."

Solution: Because in low frequencies it behaves like a resistor plus an "inductance" of value $R_1 R_2 C$, which can be a good approximation for an inductor.

(+1 point)

Have a great Winter break!