MAE143A – Signals and Systems – Fall – 2019
Final, December 13th

Instructions

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a handheld calculator with no communication capabilities.
2. You have 170 minutes.
3. Do not forget to write your name and student number.
4. This exam has 7 questions, for a total of 70 points and 1 bonus points.

Have a great Winter break!

$$\int te^{-at}dt = -\frac{(1+at)}{a^2}e^{-at}, \qquad \mathcal{L}\{t\,e^{-at}1(t)\} = \frac{1}{(s+a)^2}, \qquad \mathcal{Z}\{a^n 1[n]\} = \frac{1}{1-az^{-1}}$$

---

1. **Signal**

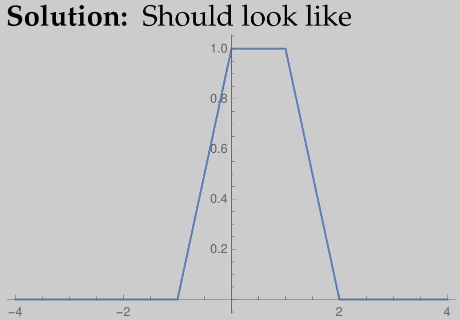
   Let

$$g(t) = \begin{cases} 0, & t < -1 \\ t+1, & -1 \le t < 0 \\ 1, & 0 \le t < 1 \\ -t+2, & 1 \le t < 2 \\ 0, & t > 2 \end{cases}$$

   Sketch the signals:

   (a) (2 points) $g(t)$

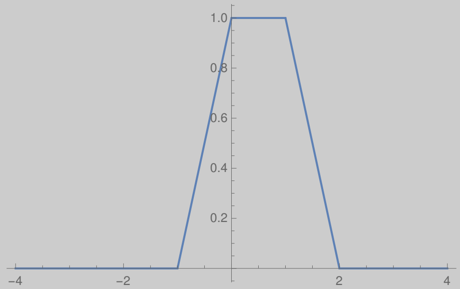
   Solution: Should look like



   (+2 points)

   (b) (2 points) $g(1-t)$

Solution: Should look like



(+2 points)

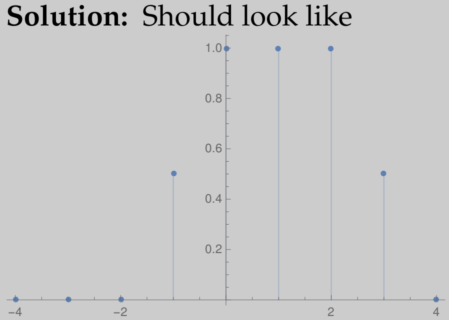
(c) (2 points) $-g(2t)$

Solution: Should look like

(+2 points)

(d) (2 points) $g(0.5n)$, $n \in \mathbb{Z}$

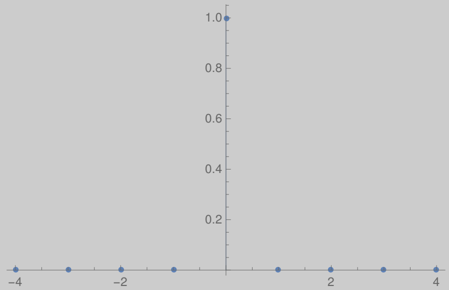
Solution: Should look like



(+2 points)

(e) (2 points) $g(2n)$, $n \in \mathbb{Z}$

Solution: Should look like



(+2 points)

2. **Fourier Series**

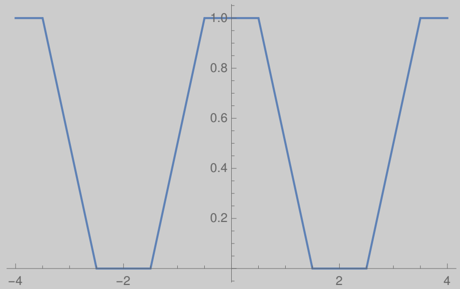
   The signal

$$g_p(t) = \sum_{k=-\infty}^{\infty} g(t - 4k + 1/2)$$

   is periodic.

   (a) (2 points) Sketch $g_p(t)$.

   Solution: Should look like



(+2 points)

   (b) (1 point) Calculate its period.

   Solution: Period is $T = 4$s because $g(t + 1/2)$ is repeated every 4s and it is zero everywhere except $-1.5 \leq t \leq 1.5$.

(+1 point)

   (c) (5 points) Show that

$$a_k = \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{8}{k^2\pi^2}\cos(\frac{k\pi}{4})\sin(\frac{k\pi}{4})^2, & k \neq 0 \end{cases}$$

   are the coefficients of the corresponding exponential Fourier Series.

   Solution: The coefficients can be calculated as

$$a_k = \frac{1}{4}\int_{-2}^{2} g(t + 1/2)\, e^{-j\omega_0 k t}\, dt, \quad \omega_0 = 2\pi/4 = \pi/2$$ (+1 point)

For $k = 0$

$$a_0 = \frac{1}{4}\int_{-2}^{2} g(t+1/2)\,dt = \frac{2}{4} = \frac{1}{2} \qquad \text{(+1 point)}$$

because the area of the graph is 2.

For $k \neq 0$

$$\begin{aligned} a_k &= \frac{1}{4}\int_{-2}^{2} g(t+1/2)\,e^{-j\omega_0 kt}\,dt \\ &= \frac{1}{4}\int_{-3/2}^{-1/2} (t+3/2)\,e^{-j\omega_0 kt}\,dt + \frac{1}{4}\int_{-1/2}^{1/2} e^{-j\omega_0 kt}\,dt + \frac{1}{4}\int_{1/2}^{3/2} (3/2-t)\,e^{-j\omega_0 kt}\,dt \end{aligned}$$

(+3 points)

NOTE TO GRADERS: From this point on is one bonus points if they get the integrals right.

Working it out

$$
\begin{aligned}
4a_k &= (3/2)\int_{-3/2}^{3/2} e^{-j\omega_0 kt} dt - (1/2)\int_{-1/2}^{1/2} e^{-j\omega_0 kt}\, dt \\
&\quad + \int_{-3/2}^{-1} t\, e^{-j\omega_0 kt}\, dt - \int_{1/2}^{3/2} t\, e^{-j\omega_0 kt}\, dt \\
&= (3/2)\frac{1}{-j\omega_0 k}(e^{-j3\omega_0 k/2} - e^{j3\omega_0 k/2}) - (1/2)\frac{1}{-j\omega_0 k}(e^{-j\omega_0 k/2} - e^{j\omega_0 k/2}) \\
&\quad + \int_{-3/2}^{-1} t\, e^{-j\omega_0 kt}\, dt - \int_{1}^{3/2} t\, e^{-j\omega_0 kt}\, dt \\
&= \frac{3\sin(3\omega_0 k/2)}{\omega_0 k} - \frac{\sin(\omega_0 k/2)}{\omega_0 k} \\
&\quad + \left.\frac{(1+j\omega_0 kt)}{(j\omega_0 k)^2} e^{-j\omega_0 kt}\right|_{-3/2}^{-1/2} - \left.\frac{(1+j\omega_0 kt)}{(j\omega_0 k)^2} e^{-j\omega_0 kt}\right|_{1/2}^{3/2} \\
&= \frac{3\sin(3\omega_0 k/2)}{\omega_0 k} - \frac{\sin(\omega_0 k/2)}{\omega_0 k} \\
&\quad + \frac{(1-j\omega_0 k/2)}{-(\omega_0 k)^2} e^{j\omega_0 k/2} - \frac{(1-j3\omega_0 k/2)}{-(\omega_0 k)^2} e^{j3\omega_0 k/2} \\
&\quad - \frac{(1+j3\omega_0 k/2)}{-(\omega_0 k)^2} e^{-j3\omega_0 k/2} + \frac{(1+j\omega_0 k/2)}{-(\omega_0 k)^2} e^{-j\omega_0 k/2} \\
&= \frac{3\sin(3\omega_0 k/2)}{\omega_0 k} - \frac{\sin(\omega_0 k/2)}{\omega_0 k} \\
&\quad + \frac{-1}{(\omega_0 k)^2}\left(e^{j\omega_0 k/2} + e^{-j\omega_0 k/2} - e^{j3\omega_0 k/2} - e^{-j3\omega_0 k/2}\right) \\
&\quad + \frac{j\omega_0 k/2}{(\omega_0 k)^2}\left(e^{j\omega_0 k/2} - e^{-j\omega_0 k/2} - 3e^{j3\omega_0 k/2} + 3e^{-j3\omega_0 k/2}\right) \\
&= \frac{3\sin(3\omega_0 k/2)}{\omega_0 k} - \frac{\sin(\omega_0 k/2)}{\omega_0 k} \\
&\quad - \frac{2}{(\omega_0 k)^2}\left(\cos(\omega_0 k/2) - \cos(3\omega_0 k/2)\right) \\
&\quad - \frac{\omega_0 k}{(\omega_0 k)^2}\left(\sin(\omega_0 k/2) - 3\sin(3\omega_0 k/2)\right)
\end{aligned}
$$

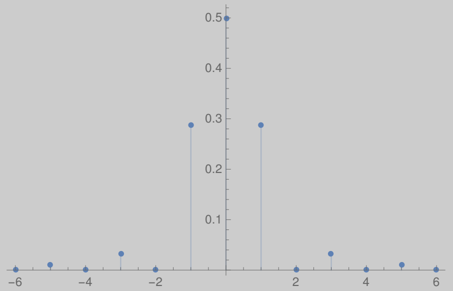
After tedious trig identities and substituting $\omega_0 = \pi/2$ one obtains

$$
a_k = \frac{4\sin(k\pi/4)\sin(k\pi/2)}{k^2\pi^2}
$$

which is equivalent to the given form.

(d) (2 points) Sketch $|a_k|$.

Solution: Should look like



(+2 points)

3. **Convolution**

   Let $g(t)$ from Question 1 be the impulse response of a time-invariant continuous-time system and answer the following questions. Show your reasoning.

   (a) (2 points) Is the system causal?

   Solution: No because $g(t) \neq 0$ for $-1 \leq t < 0$. (+2 points)

   (b) (2 points) Is the system memoryless?

   Solution: No because $y(t) = \int_{-\infty}^{\infty} g(t-\tau)u(\tau)\,d\tau$ depends on values of $u(\tau)$ for τ other than t. (+2 points)

   (c) (2 points) Is the system asymptotically stable?

   Solution: Yes because

   $$\int_{-\infty}^{\infty} |g(\tau)|\,d\tau = 2 < \infty$$

   (+2 points)

   (d) (4 points) Use the convolution formula

   $$y(t) = \int_{-\infty}^{\infty} g(t-\tau)\,u(\tau)\,d\tau$$

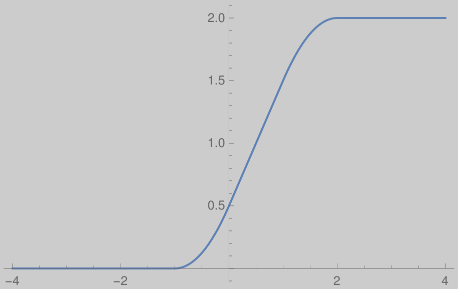
   to calculate and plot the system's step response.

Solution:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} g(t-\tau)\,1(\tau)d\tau \\
&= \int_{-\infty}^{\infty} g(\tau)\,1(t-\tau)d\tau \\
&= \int_{-\infty}^{t} g(\tau)\,1(t-\tau)d\tau \\
&= \begin{cases} 0, & t < -1 \\ \int_{-1}^{t}(t+1)dt = \frac{1}{2}t^2 + t + \frac{1}{2}, & -1 \le t < 0 \\ \frac{1}{2} + \int_0^t dt = \frac{1}{2} + t, & 0 \le t < 1 \\ \frac{3}{2} + \int_1^t(-t+2)dt = -\frac{1}{2}t^2 + 2t, & 1 \le t < 2 \\ 2, & t \ge 2 \end{cases}
\end{aligned}
$$

(+3 points)

Plot should look like



(+1 points)

NOTE TO GRADERS: A graphical solution by integrating the plot of $g(t)$ is also a possible solution.

4. **Continuous-Time System**

Consider a continuous-time system described by the differential equation

$$\ddot{y} + 2\dot{y} + y = u$$

and answer the following questions. Show your reasoning.

(a) (1 point) Is the system time-invariant?

Solution: Yes, because if $y(t)$ solve the equation for $u(t)$ then $y(t-\tau)$ solve the same equation for $u(t-\tau)$. (+1 point)

(b) (2 points) Use the Laplace transform to calculate the system's transfer-function.

Solution: Under zero initial conditions

$$s^2Y(s) + 2sY(s) + Y(s) = U(s) \qquad \text{(+1 point)}$$

after applying the Laplace transform from which

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 2s + 1}. \qquad \text{(+1 point)}$$

(c) (2 points) Is the system asymptotically stable?

Solution: Yes because the poles, that is the roots of

$$s^2 + 2s + 1 = (s+1)^2 = 0 \qquad \text{(+1 point)}$$

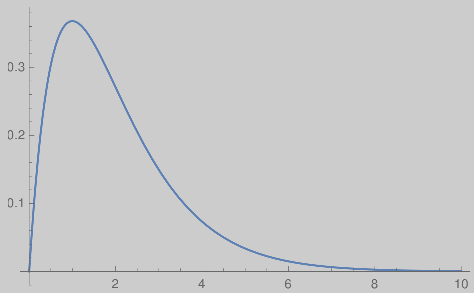
are $\{-1, -1\}$, which are on left hand side of the complex plane. (+1 point)

(d) (3 points) Calculate and sketch the system's impulse response.

Solution: The impulse response can be calculate from

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = t\,e^{-t}1(t), \qquad \text{(+1 point)}$$

a sketch of which should look like



(+1 point)

(e) (2 points) Calculate and sketch the system's response to a zero input and initial conditions $\dot{y}(0) = 1$, $y(0) = 0$.

Hint: This is simpler than you think!

Solution: Apply the Laplace transform this time with a zero input and non-zero initial conditions to obtain

$$s^2Y(s) - \dot{y}(0) - sy(0) + 2(sY(s) - y(0)) + Y(s) = 0 \qquad \text{(+1/2 point)}$$

from which

$$Y(s) = \frac{1}{s^2 + 2s + 1} = G(s) \qquad \text{(+1/2 point)}$$

hence

$$y(t) = \mathcal{L}\{G(s)\} = g(t) = te^{-t}1(t). \qquad \text{(+1 point)}$$

Plot is the same as previous question.

5. **Frequency Response**

Consider a continuous-time system described by the differential equation

$$\ddot{y} + 2\dot{y} + y = u$$

and answer the following questions.

(a) (1 point) Calculate the system's frequency response function $G(j\omega)$.

Solution: Substitue $s = j\omega$ to obtain

$$G(j\omega) = \frac{1}{(j\omega+1)^2} = \frac{1}{2j\omega + 1 - \omega^2} \qquad \text{(+1 point)}$$

(b) (3 points) Evaluate $|G(j\omega)|$ and $\angle G(j\omega)$ at $\omega = \{0, 1, \infty\}$.

Solution: Note that

$$|G(j\omega)| = \frac{1}{|j\omega+1|^2} = \frac{1}{\omega^2+1}, \qquad \angle G(j\omega) = -2\angle(j\omega+1) = -2\tan^{-1}\omega \qquad \text{(+1 point)}$$

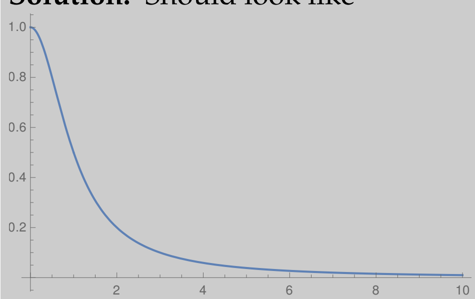
Evaluating

ω	0	1	∞
$\lvert T(j\omega)\rvert$	1	$\frac{1}{2}$	0
$\angle T(j\omega)$	0	$-\pi/2$	$-\pi$

(+2 points)

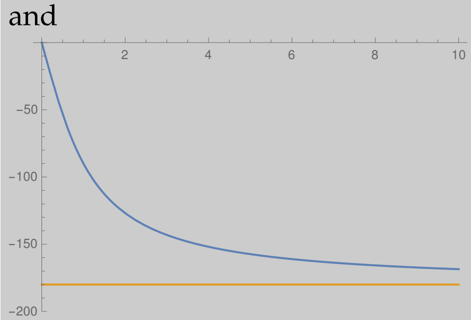
(c) (3 points) Sketch $|G(j\omega)|$ and $\angle G(j\omega)$.

Solution: Should look like



(+1.5 points)

and



(+1.5 points)

(d) (1 point) What type of filter is this system?

Solution: Low-pass filter.

(e) (2 points) Calculate the steady-state response of the system to the input

$$u(t) = \cos(t)1(t).$$

Solution: Using the frequency response

$$y_{ss}(t) = |G(j1)| \cos(t + \angle G(j1)) = \frac{1}{2}\cos(t - \pi/2) = \frac{1}{2}\sin(t).$$

6. **Discretized System**

Consider a continuous-time system described by the differential equation

$$\ddot{y} + 2\dot{y} + y = u$$

and answer the following questions. Show your reasoning.

(a) (3 points) Apply the Tustin transformation

$$s = \frac{2}{T_s}\frac{z-1}{z+1}, \qquad z = \frac{1 + sT_s/2}{1 - sT_s/2}, \qquad T_s > 0$$

to calculate the discretized transfer-function $G[z]$.

Solution: Substituting

$$\begin{aligned} G[z] &= G\left(\frac{2}{T_s}\frac{z-1}{z+1}\right) \\ &= \frac{1}{\left(\frac{2}{T_s}\frac{z-1}{z+1} + 1\right)^2} \\ &= \frac{T_s^2(z+1)^2}{(2(z-1) + T_s(z+1))^2} \\ &= \frac{T_s^2(z+1)^2}{((2+T_s)z - (2-T_s))^2} \end{aligned} \qquad \text{(+3 points)}$$

NOTE TO GRADERS: Correct answers receive full marks no matter how simplified they get.

(b) (3 points) Calculate the difference equation associated with $G[z]$.

Solution: Expanding

$$\begin{aligned} G[z] &= \frac{T_s^2(z+1)^2}{((2+T_s)z - (2-T_s))^2} \\ &= \frac{T_s^2(z^2+2z+1)^2}{(2+T_s)^2z^2 - (2+T_s)(2-T_s)z + (2+T_s-2+T_s)} \\ &= \frac{T_s^2(z^2+2z+1)}{(2+T_s)^2z^2 - 2(4-T_s^2)z + (2-T_s)^2} \\ &= \frac{T_s^2(1+2z^{-1}+z^{-2})}{(2+T_s)^2 - 2(4-T_s^2)z^{-1} + (2-T_s)^2z^{-2}} \end{aligned} \qquad \text{(+2 points)}$$

from which

$$(2+T_s)^2 y[n] - 2(4-T_s^2)y[n-1] + (2-T_s)^2 y[n-2] = T_s^2\left(u[n]+2u[n-1]+u[n-2]\right)$$
(+1 point)

NOTE TO GRADERS: A forward difference equation is also fine!

(c) (2 points) Calculate the poles of $G[z]$.

Solution: The poles are at

$$z = \left\{\frac{1-T_s/2}{1+T_s/2}, \frac{1-T_s/2}{1+T_s/2}\right\}. \qquad \text{(+2 points)}$$

This can be calculated by setting $s=-1$ on the Tustin transformation or directly from the transfer-function.

(d) (2 points) Is the discretized system asymptotically stable?

Solution: Yes, because for $T_s > 0$

$$\frac{|1-T_s/2|}{|1+T_s/2|} < 1 \qquad \text{(+2 points)}$$

(e) (1 point (bonus)) Assuming that the system will be subject to low-frequency inputs, what would you believe to be a suitable value for T_s?

Solution: Because the pole of system is at $s=-1$ is bandwidth will be limited to 1rad/s. A sampling frequency that is, say, 10 times bigger than that, say 10rad/s $\approx$ 62.8Hz, should be suitable.

7. **Discrete-Time System**

Consider a discrete-time system described by the difference equation

$$y[n] - \frac{1}{4}\,y[n-1] - \frac{1}{8}y[n-2]$$
$$= -u[n] - 2u[n-1] - u[n-2]$$

and answer the following questions. Show your reasoning.

(a) (1 point) Is the system causal?

Solution: Yes, because $y[n]$ depends only on present and past inputs and outputs.
(+1 point)

(b) (3 points) Use the Z-transform to calculate the system's transfer-function.

Solution: Applying the Z-transform

$$\left(1-\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}\right)Y[z] = -\left(1+2z^{-1}+z^{-2}\right)U[z] \qquad \text{(+2 points)}$$

from which

$$G[z] = \frac{Y[z]}{U[z]} = -\frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} \qquad \text{(+1 point)}$$

$$= -\frac{z^2 + 2z + 1}{z^2 - \frac{1}{4}z - \frac{1}{8}}$$

NOTE TO GRADERS: A transfer-function in z or z^{-1} is fine!

(c) (2 points) Calculate the transfer-function poles and zeros.

Solution: The poles are the roots of

$$z^2 - \frac{1}{4}z - \frac{1}{8} = (z - 1/2)(z + 1/4) = 0 \qquad \text{(+1 point)}$$

which are $\{-1/2, +1/4\}$.

(d) (1 point) Is the system asymptotically stable?

Solution: Yes because $\{-1/2, +1/4\}$ are both smaller than one in magnitude.

(+1 point)

(e) (3 points) Calculate and sketch the system's impulse response.

Solution: Expanding $G[z]$ in partial fractions of z^{-1} we obtain

$$\begin{aligned} G[z] &= -\frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} \\ &= 8 - \frac{9}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} \\ &= 8 - \frac{9}{(1 + \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})} \\ &= 8 - \frac{6}{1 + \frac{1}{2}z^{-1}} - \frac{3}{1 - \frac{1}{4}z^{-1}} \qquad \text{(+2 points)} \end{aligned}$$

from which

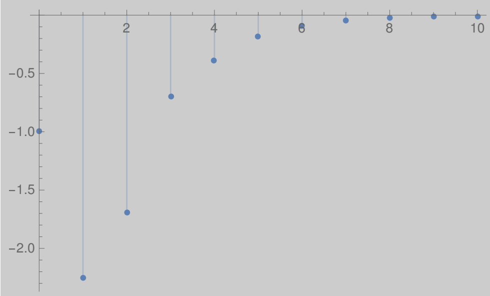
$$\begin{aligned} g[n] &= \mathcal{Z}^{-1}\{8\} - \mathcal{Z}^{-1}\left\{\frac{6}{1 + \frac{1}{2}z^{-1}}\right\} - \mathcal{Z}^{-1}\left\{\frac{3}{1 - \frac{1}{4}z^{-1}}\right\} \\ &= 8\,\delta[n] - 6\frac{1}{(2)^n} - 3\frac{1}{(-4)^n} \qquad \text{(+1 point)} \end{aligned}$$

NOTE TO GRADERS: The skecth is a bonus point!

Evaluating a couple of points

$$g[0] = 8 - 6 - 3 = -1$$
$$g[1] = -6/2 + 3/4 = -9/4$$
$$g[2] = -6/4 - 3/16 = -27/16$$
$$g[3] = -6/8 + 3/64 = -45/64$$
$$g[4] = -6/16 + 3/256 = -99/256$$
$$g[5] = -6/32 + 3/1024 = -189/1024$$

a plot should look like



(+1 bonus point)