MAE143A - Signals and Systems - Fall 2019
Midterm, October 22nd

Instructions

1. This exam is open book. You may use whatever written or printed materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.
2. You have 70 minutes.
3. Write your answers on your "Blue Book".
4. Do not forget to write your name, student number, and instructor.
5. This exam has 2 questions, for a total of 20 points.

HINT #1: Do not get stuck!
HINT #2: If a question gives you the answer needed to do another question use it!

1. **Signal Transformations and Impulse Response**

   Let the signal $g(t)$ be defined as

$$g(t) = \begin{cases} 0, & t < 0 \text{ or } t \geq 2 \\ 1, & 0 \leq t < 1 \\ -1, & 1 \leq t < 2 \end{cases}$$

   and answer the following questions:

   (a) (2 points) Sketch the signal $g(t)$;

   (b) (2 points) Sketch the signal $g(2-t)$;

   (c) (2 points) Express the signal $g(t)$ as a sum of step inputs, $1(t)$, delayed if necessary.

   Now consider a linear time-invariant continuous-time system which has as impulse response the signal $g(t)$.

   (d) (1 point) Is this system causal? Explain.

   (e) (3 points) Use convolution to calculate the response to a unit step, $u(t) = 1(t)$.

2. **Discrete-time System / Discretized System**

Consider the time-invariant discrete-time linear system

$$y[n+1] = ay[n] + b_0 u[n+1] + b_1 u[n]$$

and answer the following questions:

(a) (2 points) Show that the response of the system

$$z[n+1] = az[n] + u[n]$$

to a unit step input, $u[n] = 1[n]$, from zero initial conditions is given by

$$z[n] = (a^n - 1)/(a-1), \quad n > 0$$

HINT: use the formula $\sum_{k=0}^{n-1} a^k = (a^n - 1)(a-1)$.

(b) (2 points) Use linearity to show that the response of the original system to a unit step input, $u[n] = 1[n]$, from zero initial condition is given by

$$y[n] = (b_0 a + b_1) z[n] + b_0 u[n] = b_0 + (b_0 a + b_1)(a^n - 1)/(a-1), \quad n > 0.$$

Recall the heat equation considered in class:

$$\dot{y}(t) = -\beta \left(y(t) - u(t) \right), \quad \beta > 0$$

and answer the following questions.

(c) (2 points) A simple discretization strategy is to approximate the derivative using

$$\dot{y}(t) \approx h^{-1}(y(t+h) - y(t))$$

in which $h > 0$ is a fixed sampling time interval. Let $t = nh$ and show that the above strategy leads to the linear time-invariant discrete-time system

$$y[n+1] = ay[n] + b_1 u[n], \qquad a = (1 - h\beta), \quad b_1 = h\beta$$

(d) (2 points) Another discretization strategy is to approximate the derivative using

$$\dot{y}(t) \approx h^{-1}(y(t) - y(t-h))$$

in which $h > 0$ is a fixed sampling time interval. Let $t = nh$ and show that the above strategy leads to the linear time-invariant discrete-time system

$$y[n+1] = ay[n] + b_0 u[n+1], \qquad a = (1 + h\beta)^{-1}, \quad b_0 = h\beta(1 + h\beta)^{-1}$$

(e) (2 points) Use the answer to part (b) to calculate and compare the step responses one would obtain from the above two strategies when h is small, say $0 < h < \beta^{-1}$, or large, say $h > \beta^{-1}$. Which one would lead to better results for long running simulations?

(f) (1 point (bonus)) When h is small, then both strategies lead to system with similar behavior. Explain that by showing that the a's and b's for both strategies are indeed very similar when h is close to zero.