MAE143A - Signals and Systems - Fall 2019
Midterm, October 22nd

Instructions

1. This exam is open book. You may use whatever written or printed materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.
2. You have 70 minutes.
3. Write your answers on your “Blue Book”.
4. Do not forget to write your name, student number, and instructor.
5. This exam has 2 questions, for a total of 20 points.

HINT #1: Do not get stuck!
HINT #2: If a question gives you the answer needed to do another question use it!

1. **Signal Transformations and Impulse Response**

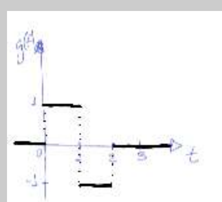
   Let the signal $g(t)$ be defined as

$$g(t) = \begin{cases} 0, & t < 0 \text{ or } t \geq 2 \\ 1, & 0 \leq t < 1 \\ -1, & 1 \leq t < 2 \end{cases}$$

   and answer the following questions:

   (a) (2 points) Sketch the signal $g(t)$;

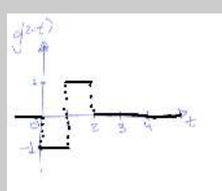
   Solution: Solution is:



(+2 points)

   (b) (2 points) Sketch the signal $g(2-t)$;

   Solution: Solution is:



(+2 points)

   (c) (2 points) Express the signal $g(t)$ as a sum of step inputs, $1(t)$, delayed if necessary.

   Solution: Solution is:

$$g(t) = u(t) - 2u(t-1) + u(t-2)$$

(+2 points)

Now consider a linear time-invariant continuous-time system which has as impulse response the signal $g(t)$.

(d) (1 point) Is this system causal? Explain.

Solution: Yes, because $g(t) = 0$ for $t < 0$. (+1 point)

(e) (3 points) Use convolution to calculate the response to a unit step, $u(t) = 1(t)$.

Solution: The response is the signal

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} g(t-\tau)u(\tau)\, d\tau \\ &= \int_{\infty}^{\infty} g(\tau)u(t-\tau)\, d\tau \\ &= \begin{cases} 0, & t < 0 \\ \int_0^t d\tau = t, & 0 \le t < 1 \\ 1 - \int_1^t d\tau = 2 - t, & 1 \le t < 2, \\ 0, & t \ge 2 \end{cases} \end{aligned} \qquad \text{(+3 points)}$$

2. **Discrete-time System / Discretized System**

Consider the time-invariant discrete-time linear system

$$y[n+1] = ay[n] + b_0 u[n+1] + b_1 u[n]$$

and answer the following questions:

(a) (2 points) Show that the response of the system

$$z[n+1] = az[n] + u[n]$$

to a unit step input, $u[n] = 1[n]$, from zero initial conditions is given by

$$z[n] = (a^n - 1)/(a-1), \quad n > 0$$

HINT: use the formula $\sum_{k=0}^{n-1} a^k = (a^n - 1)(a-1)$.

Solution: Starting from zero initial conditions calculate

$$\begin{aligned} z[1] &= u[0], \\ z[2] &= az[1] + u[1] = au[0] + u[1], \\ z[3] &= az[2] + u[2] = a^2u[0] + au[1] + u[2], \\ z[n] &= \sum_{k=0}^{n-1} a^k u[n-1-k]. \end{aligned} \qquad \text{(+1 point)}$$

For $u[n] = 1[n]$

$$z[n] = \sum_{k=0}^{n-1} a^k = \frac{a^n - 1}{a - 1} \qquad \text{(+1 point)}$$

(b) (2 points) Use linearity to show that the response of the original system to a unit step input, $u[n] = 1[n]$, from zero initial condition is given by

$$y[n] = (b_0 a + b_1)z[n] + b_0 u[n] = b_0 + (b_0 a + b_1)(a^n - 1)/(a - 1), \quad n > 0.$$

Solution: Because

$$\tilde{z}[n+1] = a\tilde{z}[n+1] + u[n+1], \quad \tilde{z}[n] = z[n+1] \qquad \text{(+1 point)}$$

use linearity to write

$$\begin{aligned} y[n] &= b_0 \tilde{z}[n] + b_1 z[n] \\ &= b_0 z[n+1] + b_1 z[n] \\ &= b_0(az[n] + u[n]) + b_1 z[n] \\ &= (b_0 a + b_1)z[n] + b_0 u[n] \\ &= b_0 + (b_0 a + b_1)(a^n - 1)/(a - 1) \qquad \text{(+1 point)} \end{aligned}$$

Recall the heat equation considered in class:

$$\dot{y}(t) = -\beta\left(y(t) - u(t)\right), \quad \beta > 0$$

and answer the following questions.

(c) (2 points) A simple discretization strategy is to approximate the derivative using

$$\dot{y}(t) \approx h^{-1}(y(t+h) - y(t))$$

in which $h > 0$ is a fixed sampling time interval. Let $t = nh$ and show that the above strategy leads to the linear time-invariant discrete-time system

$$y[n+1] = ay[n] + b_1 u[n], \qquad a = (1 - h\beta), \quad b_1 = h\beta$$

Solution: Let $t = nh$ so that

$$y(t) = y(nh) = y[n], \qquad y(t+h) = y((n+1)h) = y[n+1] \qquad \text{(+1/2 point)}$$

and write

$$h\dot{y}(t) \approx y[n+1] - y[n] = -h\beta y[n] + h\beta u[n] \qquad \text{(+1/2 point)}$$

or

$$y[n+1] = (1 - h\beta)y[n] + h\beta u[n] \qquad \text{(+1 point)}$$

(d) (2 points) Another discretization strategy is to approximate the derivative using

$$\dot{y}(t) \approx h^{-1}(y(t) - y(t-h))$$

in which $h > 0$ is a fixed sampling time interval. Let $t = nh$ and show that the above strategy leads to the linear time-invariant discrete-time system

$$y[n+1] = ay[n] + b_0 u[n+1], \qquad a = (1+h\beta)^{-1}, \quad b_0 = h\beta(1+h\beta)^{-1}$$

Solution: Let $t = nh$ so that

$$y(t) = y(nh) = y[n], \qquad y(t-h) = y((n-1)h) = y[n-1] \qquad \text{(+1/2 point)}$$

and write

$$h\dot{y}(t) \approx y[n] - y[n-1] = -h\beta y[n] + h\beta u[n] \qquad \text{(+1/2 point)}$$

that is

$$(1+h\beta)y[n] = y[n-1] + h\beta u[n]$$

or, after shifting by one unit of time and dividing by $(1+h\beta)$

$$y[n+1] = (1+h\beta)^{-1} y[n] + h\beta(1+h\beta)^{-1} u[n+1] \qquad \text{(+1 point)}$$

(e) (2 points) Use the answer to part (b) to calculate and compare the step responses one would obtain from the above two strategies when h is small, say $0 < h < \beta^{-1}$, or large, say $h > \beta^{-1}$. Which one would lead to better results for long running simulations?

Solution: For the first scheme, $a = (1-h\beta)$, $b_0 = 0$, $b_1 = h\beta$ so that from part (b):

$$\begin{aligned} y[n] &= b_0 + (b_0 a + b_1)(a^n - 1)/(a-1) \\ &= b_1(a^n - 1)/(a-1) \\ &= h\beta(a^n - 1)/(a-1), \quad a = 1 - h\beta \end{aligned} \qquad \text{(+1/2 point)}$$

NOTE TO GRADERS: answers may not be exactly in that form!
For the second scheme, $a = (1+h\beta)^{-1}$, $b_0 = h\beta(1+h\beta)^{-1}$, $b_1 = 0$ so that from part (b):

$$\begin{aligned} y[n] &= b_0 + (b_0 a + b_1)(a^n - 1)/(a-1) \\ &= b_0 + b_0 a(a^n - 1)/(a-1) \\ &= h\beta a + h\beta a^2(a^n - 1)/(a-1), \quad a = (1+h\beta)^{-1} \end{aligned} \qquad \text{(+1/2 point)}$$

NOTE TO GRADERS: answers may not be exactly in that form!
When $h < \beta^{-1}$ then $0 < 1 - h\beta < 1$ and $0 < (1+h\beta)^{-1} < 1$ and both $y[n]$ converge without oscillations. However, if $h > \beta^{-1}$ then $1 - h\beta < 0$, and the firt scheme oscillates, which is not a behavior seen in the original continuous-time system. It may not even converge at all if $1 - h\beta < -1$! The second scheme converges no matter what value of $h > 0$ is chosen because $|(1+h\beta)^{-1}| < 1$.
NOTE TO GRADERS: any reasonable comparison earns a full point. (+1 point)
Other possible comments may be concerning convergence. In the first scheme, if $|a| < 1$, then

$$\lim_{n\to\infty} y[n] = h\beta/(1-a) = 1$$

and in the second scheme

$$\lim_{n\to\infty} y[n] = h\beta a + h\beta a^2/(1-a)$$
$$= \frac{h\beta(1-a)a + h\beta a^2}{1-a} = \frac{h\beta a}{1-a} = \frac{h\beta}{a^{-1}-1} = \frac{h\beta}{1+h\beta-1} = 1$$

so both converge to the exact same solution. That solution is also the steady-state solution to the original differential equation.

(f) (1 point (bonus)) When h is small, then both strategies lead to system with similar behavior. Explain that by showing that the a's and b's for both strategies are indeed very similar when h is close to zero.

Solution: One way to show that is to linearize the expression for a and b_0 of the second scheme. Indeed, when $h \approx 0$,

$$a = (1+h\beta)^{-1} \approx 1 - h\beta$$

and

$$b_0 = h\beta(1+h\beta)^{-1} \approx h\beta$$

which are exactly the expressions for a and b_1 from the first scheme. The fact that one scheme is a function of b_0 and the other of b_1 is a consequence of the approximations be shifted by one time step.
NOTE TO GRADERS: any reasonable comparison earns a full point. (+1 bonus point)