MAE143A - Signals and Systems - Fall 2019
Midterm 2, November 21st

Instructions

1. This exam is open book. You may use whatever written or printed materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.
2. Please no visible or at reach cell phone, computer, tablet or electronic device. A visible device will get you dismissed from this exam.
3. You have 70 minutes.
4. Write your answers on your "Blue Book".
5. Do not forget to write your name, student number, and instructor.
6. This exam has 3 questions, for a total of 27 points and 2 bonus points.

HINT #1: Do not get stuck!
HINT #2: If a question gives you the answer needed to do another question use it!

1. **Fourier Transform**

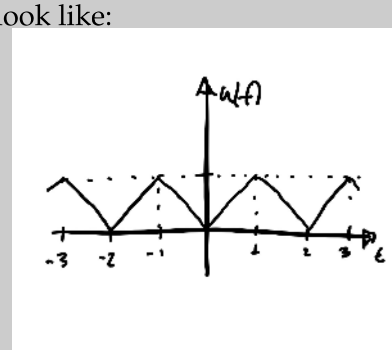
   Consider a periodic signal $u(t)$ for which one period is defined as

$$u(t) = |t|, \quad -1 \leq t \leq 1$$

   and answer the following questions:

   (a) (2 points) Sketch the signal $u(t)$.

   Solution: Solution should look like:



   (+2 points)

   NOTE TO GRADERS: Students have to plot more than one period or indicate the periodicity somehow to get two points.

   (b) (1 point (bonus)) Show that the coefficients of the exponential Fourier series of $u(t)$ are

$$a_k = \begin{cases} \frac{1}{2}, & k = 0 \\ 0, & k \neq 0 \text{ and even} \\ \frac{-2}{\pi^2 k^2}, & k \text{ odd} \end{cases}$$

Solution:
Because $T = 2$

$$\begin{aligned} a_k &= \frac{1}{2}\int_{-1}^{1} |t| e^{-j\pi k t} dt \\ &= \frac{1}{2}\int_{-1}^{0} -te^{-j\pi k t} dt + \frac{1}{2}\int_{0}^{1} te^{-j\pi k t} dt \end{aligned}$$

Knowing that

$$\int t\, e^{\alpha t} dt = \frac{\alpha t - 1}{\alpha^2} e^{\alpha t}$$

hence

$$\int_0^b t\, e^{\alpha t} dt = \frac{\alpha b - 1}{\alpha^2} e^{\alpha b} + \frac{1}{\alpha^2}$$

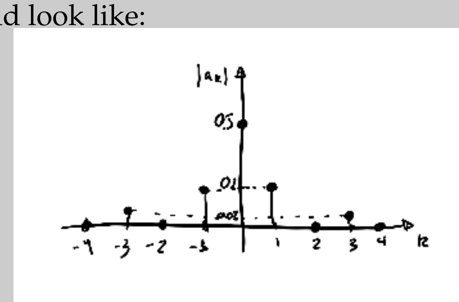
$$\begin{aligned} a_k &= \frac{1}{2}\int_{0}^{-1} te^{-j\pi k t} dt + \frac{1}{2}\int_{0}^{1} te^{-j\pi k t} dt \\ &= \frac{j\pi k - 1}{-2\pi^2 k^2} e^{j\pi k} + \frac{1}{-2\pi^2 k^2} + \frac{-j\pi k - 1}{-2\pi^2 k^2} e^{-j\pi k} + \frac{1}{-2\pi^2 k^2} \\ &= \frac{-1}{\pi^2 k^2} + \frac{j\pi k - 1}{-2\pi^2 k^2}(-1)^k + \frac{-j\pi k - 1}{-2\pi^2 k^2}(-1)^k \\ &= \frac{-1}{\pi^2 k^2} + \frac{(-1)^k}{\pi^2 k^2} \end{aligned}$$

which is the same as the provided formula for $k \neq 0$. $a_0 = 1/2$ because the area of the curve is equal to 1 and the period is 2.

(+2 points)

(c) (2 points) Sketch the line spectrum $|a_k|$ of $u(t)$ as a function of k.

Solution: Solution should look like:



(+2 points)

(d) (2 points) Use part (b) to calculate the Fourier transform of the signal $u(t)$.

Solution: Because

$$u(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\pi k t}$$

and the Fourier transform of $e^{j\omega_0 t}$ is equal to $\delta(\omega - \omega_0)$... (+1 point)
... the Fourier transform of $u(t)$ is

$$U(\omega) = \sum_{k=-\infty}^{\infty} a_k \delta(\omega - \pi k).$$ (+1 point)

2. **Frequency-domain Filtering**

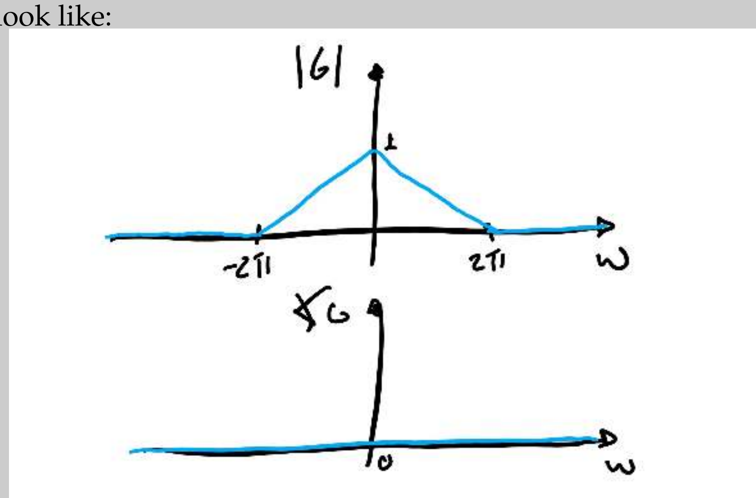
Consider a linear time-invariant filter defined by the frequency response function

$$G(j\omega) = \begin{cases} 1 - |\omega|/(2\pi) & |\omega| \leq 2\pi, \\ 0 & |\omega| \geq 2\pi, \end{cases}$$

and answer the following questions:

(a) (3 points) Sketch the magnitude $|G(j\omega)|$ and phase $\angle G(j\omega)$ of the frequency response.

Solution:
Note that $\angle G(j\omega) = 0$ because $G(j\omega)$ is real and nonnegative. (+1 point)
Plots should look like:



(+2 points)

(b) (1 point) What type of filter is G? E.g low-pass, high-pass, etc.

Solution: Low pass, since it has a zero response to frequencies higher than $|\omega| > 2\pi$. (+1 point)

(c) (1 point (bonus)) Show that the impulse response of the above filter is given by

$$g(t) = \frac{\sin(\pi t)^2}{\pi^2 t^2}.$$

Solution: One solution is via brute force:

$$\begin{aligned}
g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} (1-|\omega|/(2\pi))e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-2\pi}^{0} (1+\omega/(2\pi))e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{0}^{2\pi} (1-\omega/(2\pi))e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{0}^{2\pi} (1-\omega/(2\pi))e^{j\omega t}d\omega - \frac{1}{2\pi}\int_{0}^{-2\pi} (1+\omega/(2\pi))e^{j\omega t}d\omega
\end{aligned}$$

Using the same formula as in Prob. 1

$$\int (1-\omega/b)\, e^{j\omega t}d\omega = \frac{e^{j\omega t}}{jt} - \frac{j\omega t-1}{-bt^2}e^{j\omega t} = \frac{1+jt(b-\omega)}{-bt^2}e^{j\omega t}$$

and

$$\int_0^b (1-\omega/b)\, e^{j\omega t}d\omega = \frac{1}{-bt^2}e^{jtb} - \frac{1+jtb}{-bt^2},$$

$$\begin{aligned}
\int_0^b (1-\omega/b)\, e^{j\omega t}d\omega - \int_0^{-b} (1+\omega/b)\, e^{j\omega t}d\omega & \\
&= \frac{1}{-bt^2}e^{jtb} - \frac{1+jtb}{-bt^2} - \frac{1}{bt^2}e^{-jtb} + \frac{1-jtb}{bt^2} \\
&= \frac{2}{bt^2} - \frac{1}{bt^2}(e^{jtb}+e^{-jtb}) \\
&= \frac{2}{bt^2}(1-\cos(tb)) \\
&= \frac{4}{bt^2}\sin(tb/2)^2
\end{aligned}$$

from which one obtains

$$g(t) = \frac{1}{2\pi}\frac{4}{2\pi t^2}\sin(t2\pi/2)^2 = \frac{\sin(\pi t)^2}{\pi^2 t^2}.$$

An alternative solution is to recognize that the given spectrum is the convolution of two ideal lowpass filters with cutoff frequency π. That is

$$G(\omega) = \frac{1}{2\pi}LP(\omega) * LP(\omega)$$

from which

$$g(t) = 2\pi LP(t)LP(t) = \frac{\sin(\pi t)^2}{\pi^2 t^2}.$$

(d) (1 point) Is this filter causal? Explain.

Solution: No because $g(t) \neq 0$ for $t < 0$. (+1 point)

(e) (2 points) Is this filter asymptotically stable? Explain.
Hint: $|g(t)| \leq 1/(\pi t)^2$, $|t| \geq 1/\pi$ *and* $|g(t)| \leq 1$, $|t| \leq 1/\pi$.

Solution: Yes because

$$\begin{aligned}\int_{-\infty}^{\infty} |g(t)| dt &\leq 2 \int_{-1/\pi}^{1/\pi} dt + 2 \int_{1/\pi}^{\infty} \frac{1}{(\pi t)^2} dt \\ &= \frac{4}{\pi} + 2 \left. \frac{-1}{\pi^2 t} \right|_{1/\pi}^{\infty} \\ &= \frac{4}{\pi} + \frac{2}{\pi} = \frac{6}{\pi}\end{aligned}$$

which is bounded. (+1 point)

(f) (2 points) Calculate the steady-state response of the filter to the periodic input $u(t)$ from Problem 1.

Solution:
Since

$$U(\omega) = \sum_{k=-\infty}^{\infty} a_k \delta(\omega - \pi k).$$

the response is given by

$$Y(\omega) = G(\omega) U(\omega) = \sum_{k=-1}^{1} a_k (1 - |\omega|/(2\pi)) \delta(\omega - \pi k). \qquad \text{(+1 point)}$$

because $G(\omega) = 0$ for $|\omega| \geq 2\pi$ and $a_2 = a_{-2} = 0$, from which

$$\begin{aligned}y(t) &= a_0 + a_1 (1 - |\pi|/(2\pi)) e^{j\pi t} + a_{-1} (1 - |\pi|/(2\pi)) e^{-j\pi t} \\ &= \frac{1}{2} - \frac{1}{2} \frac{2}{\pi^2} (e^{j\pi t} + e^{-j\pi t}) \\ &= \frac{1}{2} - \frac{2}{\pi^2} \cos(\pi t)\end{aligned} \qquad \text{(+1 point)}$$

3. **Time-domain Filtering**

Consider the linear time-invariant system defined by the ODE

$$\dot{y} = -y + u - \dot{u}$$

and answer the following questions:

(a) (2 points) Assume zero initial-conditions and use the Laplace or the Fourier transform to calculate the system's transfer-function $G(s)$ or $G(j\omega)$.

Solution: For example, with the Laplace transform:

$$sY(s) = -Y(s) + U(s) - sU(s) \qquad \text{(+1 point)}$$

from which

$$Y(s) = G(s)U(s), \qquad G(s) = \frac{1-s}{1+s} \qquad \text{(+1 point)}$$

NOTE TO GRADERS: A Fourier approach would lead to

$$G(j\omega) = \frac{1-j\omega}{1+j\omega}.$$

(b) (3 points) Sketch the magnitude $|G(j\omega)|$ and phase $\angle G(j\omega)$ of the frequency response.

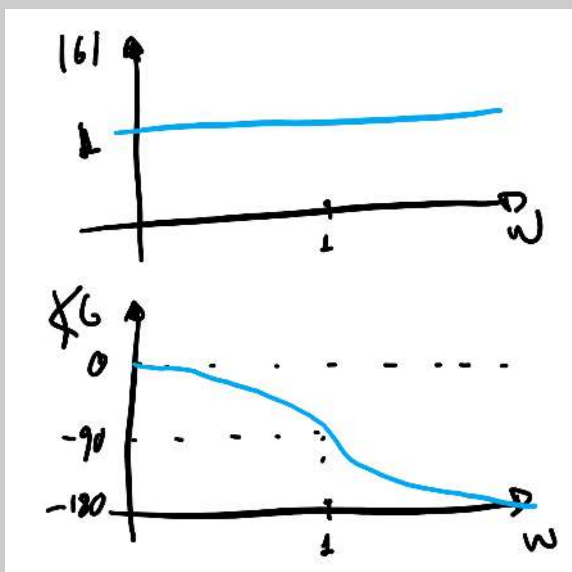
Solution: Note that

$$|G(j\omega)| = \frac{|1-j\omega|}{|1+j\omega|} = \frac{\sqrt{1+\omega^2}}{\sqrt{1+\omega^2}} = 1 \qquad \text{(+1 point)}$$

and

$$\angle G(j\omega) = \angle 1 - j\omega - \angle 1 + j\omega = \tan^{-1} -\omega - \tan^{-1}\omega = -2\tan^{-1}\omega \qquad \text{(+1 point)}$$

for which the plots should look like:



(+1 point)

(c) (1 point) The type of filter of the above G is known as *all-pass*. Can you explain why?

Solution: Because $|G(j\omega)| = 1$ for all ω, all frequencies will "pass," therefore all-pass. (+1 point)

(d) (2 points) Use the inverse Laplace or Fourier transform to calculate the impulse response $g(t)$.

Solution: The inverse Laplace transform of

$$G(s) = \frac{1-s}{1+s} = \frac{2}{s+1} - 1 \qquad \text{(+1 point)}$$

is

$$g(t) = 2e^{-t}1(t) - \delta(t). \qquad \text{(+1 point)}$$

(+1 point)

(e) (1 point) Is this system causal? Explain.

Solution: Yes because $g(t) = 0$ for all $t < 0$. (+1 point)

(f) (1 point) Is this system asymptotically stable? Explain.

Solution: Yes because $G(s)$ has only one pole at -1, which has negative real part. (+1 point)

(g) (2 points) Calculate the the steady-state response of the system to the periodic input $u(t) = \cos(t)$.

Solution: The steady-state response is given by

$$y_{ss}(t) = |G(j\omega)| \cos(\omega t + \angle G(j\omega)) \qquad \text{(+1 point)}$$

which, for $\omega = 1$ evaluates to

$$y_{ss}(t) = \cos(t - \pi/2) = \sin(t) \qquad \text{(+1 point)}$$

because $\angle G(j1) = -2\tan^{-1} 1 = -\pi/2$.