MAE143A - Signals and Systems - Fall 2020
Final, December 15th

Instructions

1. You have 2 hours to finish this exam.
2. You have until 10:10AM to submit your answers on gradescope.
3. The exam is to be done individually.
4. All answers should be manuscript.
5. No typed answers of any form will be accepted.
6. All graphs and diagrams have to be sketched by hand. Include enough detail.
7. If you use a tablet to do the work make sure it is handwritten.
8. This exam has 3 questions, for a total of 30 points.

1. **Signals and Transforms**

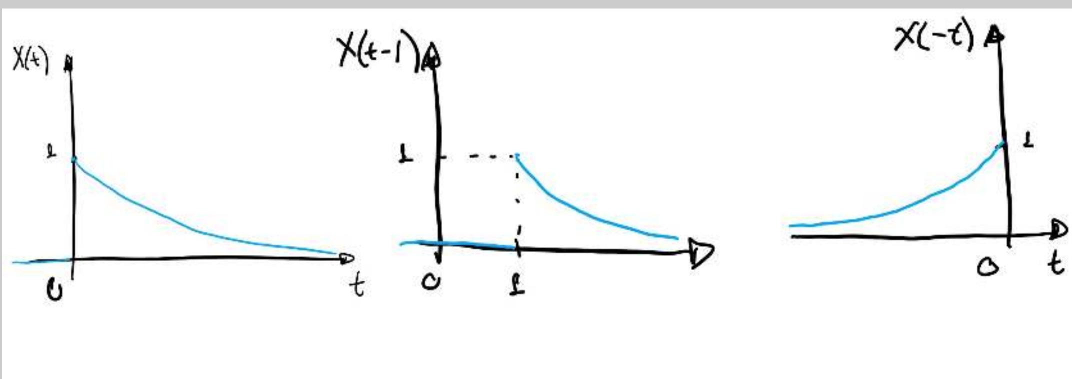
   Consider the continuous-time signal

$$x(t) = 2^{-t}\,1(t)$$

   and answer the following questions.

   (a) (3 points) Sketch the signals $x(t)$, $x(t-1)$, $x(-t)$.

   Solution:
   Sketches should look like:



   (+3 points)

   (b) (1 point) What is the Laplace transform of $x(t)$ and its region of convergence?

   Hint #1: calculate a such that $2^{-t} = e^{-at}$.
   Hint #2: answer from a table is fine!

Solution:
From a table

$$X(s) = \mathcal{L}\{2^{-t}1(t)\} = \mathcal{L}\{e^{-\log(2)t}1(t)\} = \frac{1}{s+\log(2)}, \quad \mathrm{Re}\{s\} > -\log(2).$$

(+1 point)

NOTE TO GRADERS: In this and following questions, it does not matter if students give the answer as 2^{-t} or $e^{-\log(2)t}$.

(c) (1 point) Can you calculate the Fourier transform of $x(t)$ from its Laplace transform? Explain.

Solution:
Yes, because the region of convergence contains the imaginary axis (or $X(s)$ is asymptotically stable) $\mathcal{F}\{x(t)\} = X(s = j\omega)$. (+1 point)

(d) (1 point) Calculate the generalized derivative $\dot{x}(t)$.

Hint: calculate a such that $2^{-t} = e^{-at}$.

Solution:

$$\dot{x}(t) = \delta(t) - \log(2)e^{-\log(2)t}1(t) = \delta(t) - \log(2)2^{-t}1(t)$$

(+1 point)

(e) (1 point) Calculate $\dot{x}(t)$ using the Laplace transform.

Hint: calculate $\dot{x}(t) = \mathcal{L}^{-1}\{?\}$

Solution:

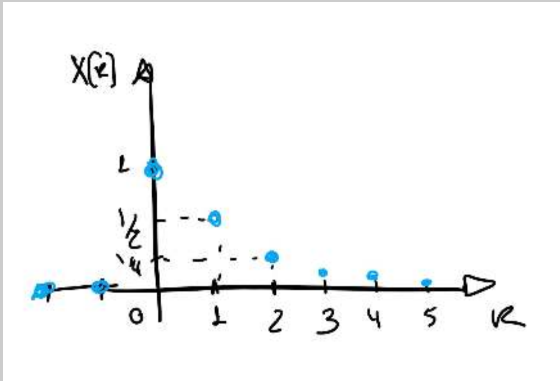
$$\dot{x}(t) = \mathcal{L}^{-1}\left\{\frac{s}{s+\log(2)}\right\} = \mathcal{L}^{-1}\left\{1 - \frac{\log(2)}{s+\log(2)}\right\} =$$

$$\delta(t) - \log(2)e^{-\log(2)t}\,1(t) = \delta(t) - \log(2)2^{-t}\,1(t).$$

(+1 point)

(f) (1 point) Sketch the signal $x[k] = x(k)$.

Solution:
Sketch should look like:



(+1 point)

(g) (1 point) What is the z-transform of $x[k] = x(k)$?

Hint: answer from a table is fine!

Solution:
From a table

$$X(z) = \mathcal{Z}\{2^{-k}\,1[k]\} = \frac{z}{z-2}, \quad |z| > 2$$ (+1 point)

2. **Continuous-time System**

The velocity of a vehicle traveling on a straight line, $y(t)$, is given by the following ODE:

$$m\,\dot{y}(t) + b\,y(t) = u(t)$$

in which $m > 0$ is the vehicle's mass, $b > 0$ is a coefficient of viscous friction, and $u(t)$ is an external input force.

(a) (2 points) Calculate the system's transfer-function $G(s)$ and its region of convergence.

Solution:
By inspection

$$G(s) = \frac{1/m}{s + b/m}$$ (+1 point)

Because the system is causal the region of convergence is

$$\mathrm{Re}\{s\} > -b/m.$$ (+1 point)

(b) (1 point) Calculate the transfer-function's poles and zeros.

Solution:
No zeros, pole is:

$$s = -b/m. \qquad \text{(+1 point)}$$

(c) (1 point) Calculate the system's impulse response.

Solution:

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{1}{m} e^{-(b/m)t}\, 1(t). \qquad \text{(+1 point)}$$

(d) (1 point) Is the system asymptotically stable? Explain.

Solution:
Yes, the pole $s = -b/m$ is negative.

(e) (3 points) Calculate the system's response to a unit step force input. Do not assume zero initial conditions.

Solution:
Transform of the input is:

$$U(s) = \frac{1}{s} \qquad \text{(+1/2 point)}$$

Response is

$$Y(s) = G(s)U(s) + \frac{x_1(0)}{s + b/m} = \frac{1/m}{s + b/m}\frac{1}{s} + \frac{y(0^+)}{s + b/m} \qquad \text{(+1/2 point)}$$

because the initial state is equal to the initial condition.

$$\begin{aligned} Y(s) &= \frac{1/m}{s + b/m}\frac{1}{s} + \frac{y(0^+)}{s + b/m} \\ &= \frac{1}{b}\left(\frac{1}{s} - \frac{1}{s + b/m}\right) + \frac{y(0^+)}{s + b/m} \end{aligned} \qquad \text{(+1 point)}$$

and

$$y(t) = \frac{1}{b}\left(1 - e^{-(b/m)t}\right) 1(t) + y(0^+)\, e^{-(b/m)t}\, 1(t) \qquad \text{(+1 point)}$$

(f) (3 points) Let $m = b = 1$, calculate $|G(j\omega)|$ and $\angle G(j\omega)$. Evaluate $|G(j\omega)|$ for $\omega = \{0, 1, \infty\}$, and sketch the magnitude of the frequency response function $G(j\omega)$.

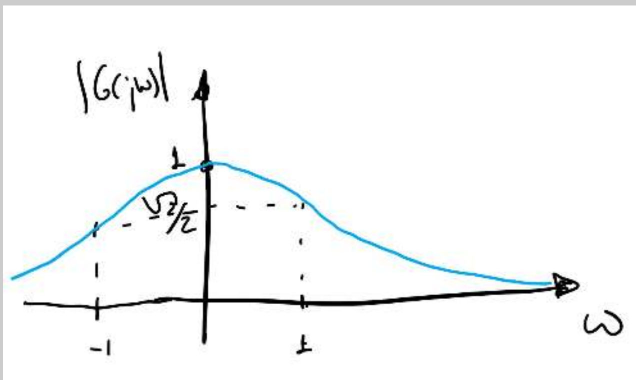
Solution:

$$|G(j\omega)| = \frac{1}{|j\omega + 1|} = \frac{1}{\sqrt{\omega^2 + 1}}, \qquad \angle G(j\omega) = -\tan^{-1}\omega \qquad \text{(+1 point)}$$

Values of the magnitude are

$$|G(j0)| = 1, \qquad |G(j1)| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \qquad |G(j\infty)| = 0. \qquad \text{(+1 point)}$$

Sketch should look like:



(+1 point)

(g) (2 points) Let $m = b = 1$, and use the frequency response to calculate the steady-state component of the response to an input of the form

$$u(t) = \cos(t)$$

Explain why is this possible.

Hint: use answer from previous question.

Solution:

$$y(t) = \frac{1}{\sqrt{1+1}} \cos(\omega t - \tan^{-1} 1) = \frac{\sqrt{2}}{2} \cos(\omega t - \pi/4) \qquad \text{(+1 point)}$$

It is possible because the system is asymptotically stable. (+1 point)

3. **Discrete-time System**

A discrete-time system is modeled by the difference equation

$$y[k+1] = u[k+1] - u[k]$$

(a) (2 points) Calculate the system's transfer-function $G(z)$ and its region of convergence.

Solution: By inspection from the difference equation

$$G(z) = \frac{z-1}{z}. \qquad \text{(+1 point)}$$

Its region of convergence is $z \neq 0$ ($|z| > 0$). (+1 point)

(b) (1 point) Calculate the transfer-function's poles and zeros.

Solution:
Zero at

$$z = 1 \qquad \text{(+1/2 point)}$$

and pole at

$$z = 0. \qquad \text{(+1/2 point)}$$

(c) (1 point) Is the system causal? Explain.

Solution: Yes because $G(z)$ is proper. (+1 point)

(d) (1 point) Is the system asymptotically stable? Explain.

Solution: Yes because the pole $z = 0$ is inside the unit circle. (+1 point)

(e) (3 points) Assume zero-initial state and calculate the response of the system to a unit step input.

Solution:
Transform of the input is:

$$U(z) = \frac{z}{z-1} \qquad \text{(+1 point)}$$

Response is

$$Y(z) = G(z)U(z) = \frac{z-1}{z}\frac{z}{z-1} = 1 \qquad \text{(+1 point)}$$

from which

$$y[k] = \delta[k] \qquad \text{(+1 point)}$$