MAE143A - Signals and Systems - Fall 2020
Midterm, October 22nd

Instructions

1. You have until Thursday at 8:00PM to submit your answers on gradescope.
2. The exam is to be done individually.
3. All answers should be manuscript.
4. No typed answers of any form will be accepted.
5. All graphs and diagrams have to be sketched by hand. Include enough detail.
6. If you use a tablet to do the work make sure it is handwritten.
7. This exam has 4 questions, for a total of 36 points and 1 bonus points.

Notes on Academic integrity

1. Unlikely answers that look similar will be flagged as cheating. Your grade might be impacted and you may be reported.
2. We will complement the exam with an oral exam for a randomly selected number of students. It will work like this:
   - Next Wednesday, October 28th, during discussion session all students have to be present at the beginning of the session.
   - A number of students will be randomly selected for oral interview to take place during discussion session. It is mandatory that all students attend the beginning of the discussion. If your name is selected and you are not available for interview you will immediately receive a zero on the midterm. If you have any problems attending this session please contact me BEFORE next Wednesday so that we can work out an alternative.
   - In this in person (zoom) interview you will have to provide oral arguments to support the solutions that you turned in.
   - If you cannot successfully support your written answer your grade will be impacted and you might be reported for cheating.
3. For your information, with the numbers I am working on as of now, if 15 students cheat in the exam there is a 95% chance that at least one of them will be interviewed. If 20 students cheat, the chance is 98%, and if 25 cheat, 99.6%. You get the idea, right?

1. **Signal Transformations**

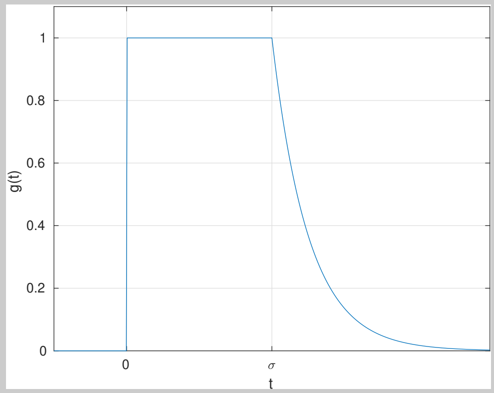
Consider a signal

$$g(t) = 1(t) - (1 - e^{-2(t-\sigma)})\, 1(t-\sigma), \quad \sigma > 0$$

and answer the following questions:

(a) (3 points) Sketch the signal $g(t)$;

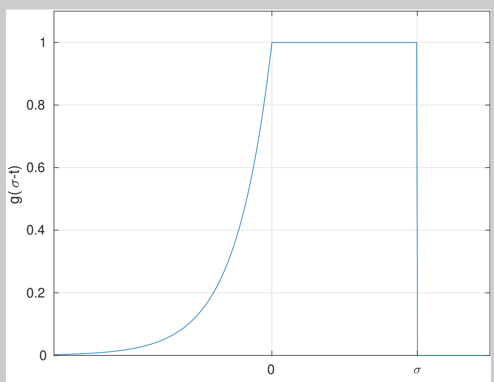
Solution: Solution is:



(+3 points)

(b) (3 points) Sketch the signal $g(\sigma - t)$;

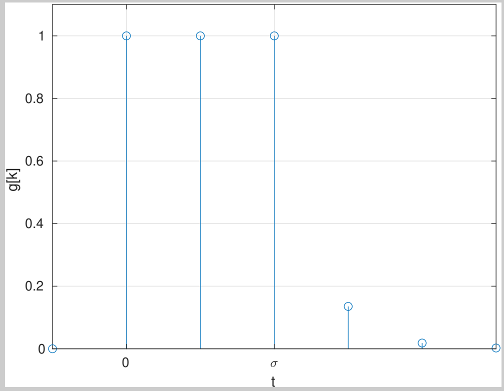
Solution: Solution is:



(+3 points)

(c) (3 points) Sketch the signal $g[k] = g(kT)$, $T = \sigma/2$;

Solution: Solution is:



(+3 points)

NOTE: Make sure your sketches contain all information needed to clearly identify the signals.

2. **Continuous-Time Impulse Response**

   Consider a linear time-invariant continuous-time system which has as impulse response the signal $g(t)$ from Question 1.

   (a) (2 points) Is this system causal? Explain.

   Solution: Yes, because $g(t) = 0$ for $t < 0$. (+2 points)

   (b) (4 points) Use convolution to calculate the response to a unit step, $u(t) = 1(t)$. Sketch the response.

   Solution: The response is the signal

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} g(\tau)\, 1(t-\tau)\, d\tau \\ &= \int_0^t g(\tau)\, d\tau \\ &= \begin{cases} 0, & t < 0 \\ \int_0^t d\tau = t, & 0 \le t < \sigma \\ \sigma + \int_\sigma^t (1 - 1 + e^{-2(\tau-\sigma)})\, d\tau, & t \ge \sigma \end{cases} \end{aligned}$$

(+2 points)

The last integral evaluates to

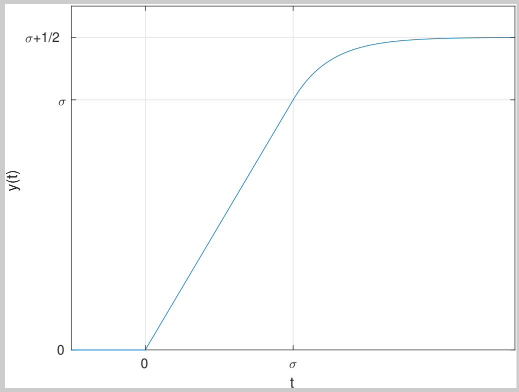
$$\begin{aligned} \int_\sigma^t e^{-2(\tau-\sigma)}\, d\tau &= \int_0^{t-\sigma} e^{-2\zeta}\, d\zeta, \quad \zeta = \tau - \sigma \\ &= \frac{1}{-2} e^{-2\zeta} \Big|_{\zeta=0}^{t-\sigma} \\ &= \frac{1}{2} - \frac{1}{2} e^{-2(t-\sigma)} \end{aligned}$$

so that

$$y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \le t < \sigma \\ \frac{1}{2} + \sigma - \frac{1}{2}e^{-2(t-\sigma)}, & t \ge \sigma \end{cases}$$

(+1 point)

Sketch is:



(+1 points)

(c) (3 points) Is the system BIBO stable? Show your work.

Solution: Because $g(t) \ge 0$

$$\int_0^\infty |g(\tau)|\, d\tau = \lim_{t\to\infty} \int_0^t g(\tau)\, d\tau = \lim_{t\to\infty} \int_0^t g(\tau) 1(t-\tau)\, d\tau$$

which you already calculated in part (b). (+1 point)

That is

$$\int_0^\infty |g(\tau)|\, d\tau = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{1}{2} + \sigma - \frac{1}{2}e^{-2(t-\sigma)} = \frac{1}{2} + \sigma$$

(+1 point)

Since this is bounded the system is BIBO stable. (+1 point)

3. **LTI System Modeled by an ODE**

An LTI system is modelled by the ODE

$$\dot{y}(t) + y(t) = u(t)$$

(a) (2 points) Calculate the associated transfer-function $G(s)$.

Solution: The transfer-function can be calculated by inspection from the coefficients of the ODE as

$$G(s) = \frac{1}{s+1}$$

(+2 points)

(b) (3 points) Based on the transfer-function is the LTI system BIBO stable?

Solution: The pole of the transfer-function is $s = -1$ so that $G(s)$ converges for all $\mathrm{Re}\{s\} > -1$ which includes the entire right-half plane so the system is BIBO stable. (+3 points)

(c) (4 points) Use the transfer-function from part (a) to calculate the response of the system to the input $u(t) = \cos(t)$. Explain the assumptions needed in your approach.

Solution: From the frequency response formula

$$y(t) = |G(j\omega)| \cos(\omega t + \angle G(j\omega)), \qquad G(j\omega) = \frac{1}{j\omega + 1}. \qquad \text{(+1 point)}$$

Substituting $\omega = 1$

$$|G(j1)| = \frac{1}{|j+1|} = \frac{\sqrt{2}}{2}, G(j1) = -\tan^{-1} 1 = -\frac{\pi}{4} \qquad \text{(+1 point)}$$

so that the response is

$$y(t) = \frac{\sqrt{2}}{2} \cos(t - \pi/4) \qquad \text{(+1 point)}$$

For using the frequency response formula the system needs to be asymptotically stable, which is. See part (b). (+1 point)

(d) (1 point (bonus)) Use the transfer-function from part (a) to calculate the response of the system to the input $u(t) = \sin(t)$.

Solution: Similar to part (c) but note that

$$u(t) = \sin(t) = \cos(t - \pi/2)$$

and the frequency response formula

$$y(t) = |G(j\omega)| \cos(\omega t - \pi/2 + \angle G(j\omega)), \qquad G(j\omega) = \frac{1}{j\omega + 1}.$$

so that the response is

$$y(t) = \frac{\sqrt{2}}{2} \cos(t - 3\pi/4)$$

(+1 bonus point)

4. **Discrete-time System**

A discrete-time system is described by the recursion

$$y[k+1] - y[k] = u[k]$$

(a) (3 points) Calculate the system's impulse response, $g[k]$.

Solution: Set $u[k] = \delta[k]$ to see that

$$\begin{aligned} y[0] &= 0, \\ y[1] &= u[0] = 1, \\ y[2] &= y[1] + u[1] = 1, \\ &\vdots \quad \vdots \\ y[k] &= 1. \end{aligned}$$

(+2 points)

That is $g[k] = 1[k-1]$. (+1 points)

(b) (3 points) Use the recursion to calculate the system's response to a step input $u[k] = 1[k]$ and zero initial condition $y[0] = 0$.

Solution: Set $u[k] = 1[k]$ to see that

$$\begin{aligned} y[0] &= 0, \\ y[1] &= u[0] = 1, \\ y[2] &= y[1] + u[1] = 1 + 1 = 2, \\ y[3] &= y[2] + u[2] = 2 + 1 = 3, \\ &\vdots \quad \vdots \\ y[k] &= k. \end{aligned}$$

(+2 points)

That is $g[k] = k\,1[k]$. (+1 points)

(c) (3 points) Repeat part (b) using the impulse response and a discrete-time convolution.

Solution: Using convolution with $g[k] = u[k] = 1[k-1]$,

$$y[k] = \sum_{\ell=-\infty}^{\infty} g[\ell]\, u[k-\ell] = \sum_{\ell=-\infty}^{\infty} 1[\ell-1]\, 1[k-\ell] = \sum_{\ell=1}^{k} 1 = k\,1[k]$$

(+3 points)

which agrees with part (b).