MAE143A - Signals and Systems - Fall 2020
Midterm 2, November 19th

Instructions

1. You have until Thursday at 10:00PM to submit your answers on gradescope.
2. The exam is to be done individually.
3. All answers should be manuscript.
4. No typed answers of any form will be accepted.
5. All graphs and diagrams have to be sketched by hand. Include enough detail.
6. If you use a tablet to do the work make sure it is handwritten.
7. This exam has 4 questions, for a total of 40 points and 1 bonus points.

Notes on Academic integrity

1. Unlikely answers that look similar will be flagged as cheating. Your grade might be impacted and you may be reported.
2. We will complement the exam with an oral exam for a randomly selected number of students. It will work like this:
   - Next Wednesday, November 25th, during discussion session all students have to be present at the beginning of the session.
   - A number of students will be randomly selected for oral interview to take place during discussion session. It is mandatory that all students attend the beginning of the discussion. If your name is selected and you are not available for interview you will immediately receive a zero on the midterm. If you have any problems attending this session please contact me BEFORE next Wednesday so that we can work out an alternative.
   - In this in person (zoom) interview you will have to provide oral arguments to support the solutions that you turned in.
   - If you cannot successfully support your written answer your grade will be impacted and you might be reported for cheating.
3. For your information, with the numbers I am working on as of now, if 15 students cheat in the exam there is a 95% chance that at least one of them will be interviewed. If 20 students cheat, the chance is 98%, and if 25 cheat, 99.6%. You get the idea, right?

1. **Continuous-time System Response**

   A continuous-time linear time-invariant system has as impulse response

   $$g(t) = \delta(t) - 10\, e^{-10\, t} 1(t).$$

   (a) (3 points) Calculate the system's transfer-function $G(s)$ and its region of convergence. *Hint: use any method you would like.*

   (b) (1 point) Write the ordinary differential equation corresponding to the system.

   (c) (1 point) Is the system causal? Explain.

   (d) (1 point) Is the system asymptotically stable? Explain.

   (e) (4 points) Use the Fourier transform to calculate the output response of the system, $y(t)$, to the input

   $$u(t) = \mathrm{rect}(t).$$

   Sketch the response.

2. **Discrete-time System Response**

   A discrete-time system is modeled by the difference equation

   $$y[k+1] = a\, y[k] + (1-a)\, u[k], \quad 0 \le a < 1.$$

   (a) (2 points) Calculate the system's transfer-function $G(z)$ and its region of convergence.

   (b) (1 point) Is the system causal? Explain.

   (c) (1 point) Is the system asymptotically stable? Explain.

   (d) (3 points) Assume zero-initial conditions and calculate the response of the system to a unit step input. *Hint: It is OK to construct the solution iteratively.*

   (e) (2 points) Calculate the magnitude and phase of the system's frequency response $G(e^{j\omega})$.

   (f) (1 point) Sketch the magnitude of the frequency response.

3. **Frequency Response**

   Consider a continuous-time LTI system with transfer-function $G(s)$. Assume that $G(s)$ is asymptotically stable and with real coefficients.

   (a) (1 point) Is $G(j\omega)$ the Fourier transform of the impulse response? Explain.

   (b) (2 points) Calculate the Fourier transform of the input signal $u(t) = \cos(\sigma t + \phi)$.

   (c) (2 points) Calculate the Fourier transform, $Y(j\omega)$, of the output signal $y(t)$ if $u(t) = \cos(\sigma t + \phi)$ is the input.

   (d) (4 points) Calculate the output response $y(t)$ by calculating the inverse Fourier transform of $Y(j\omega)$ from part (c).

   (e) (1 point) Compare your answer from part (d) with the answer you would have obtained using the frequency response method.

4. **Amplifier**

In response to an input signal $u(t)$ an amplifier produces the output

$$y(t) = \begin{cases} 1, & u(t) \geq 1/2\ , \\ 2\,u(t), & |u(t)| < 1/2\ , \\ -1, & u(t) \leq -\ 1/2\ . \end{cases}$$

Consider a periodic input signal

$$u(t) = \cos(t)$$

and answer the following questions:

(a) (2 points) Determine the range of t in which $|u(t)| \geq 1/2$ and use that to write a piecewise description for the signal $y(t)$. Sketch the signals $u(t)$ and $y(t)$.

(b) (1 point) Let

$$y_d(t) = 2\,u(t) - y(t)$$

be the *distortion signal*. Sketch $y_d(t)$.

(c) (2 points) Show that the coefficients of the exponential Fourier series of $y_d(t)$ for $n = 0$ and $n = 1$ are

$$\hat{Y}_d[0] = 0, \qquad\qquad \hat{Y}_d[1] = \frac{2}{3} - \frac{\sqrt{3}}{2\pi}.$$

(d) (1 point (bonus)) Calculate all other coefficients of the exponential Fourier series of $y_d(t)$.

(e) (1 point) Without calculating any integrals, show how the coefficients of the exponential Fourier series of $y_d(t)$ can be used to calculate the coefficients of the Fourier series of $y(t)$. *Hint: You do not need to have calculated the coefficients to answer this question!*

(f) (2 points) The average power of a periodic signal is

$$P_x = \frac{1}{T_0} \int_0^{T_0} x(t)^2\,dt.$$

Calculate the average power of $u(t)$ and $y_d(t)$.
Hint:

$$\int \cos(t)^2\,dt = \frac{t}{2} + \frac{1}{4}\sin(2t).$$

(g) (2 points) Use question (e) and Parseval's theorem to show that

$$P_y = P_{y_d} + 2(1 - 2\,\mathrm{Re}\{\hat{Y}_d[1]\}),$$

in which $\hat{Y}_d[1]$ is the coefficient $n = 1$ from item (c). Calculate P_y using parts (c) and (f).

	$x(t)$		$X(j\omega)$
impulse	$\delta(t)$		1
	$\delta^{(n)}(t)$		$(j\omega)^n$
constant	1		$2\pi\,\delta(\omega)$
step	$1(t)$		$\dfrac{1}{j\omega} + \pi\,\delta(\omega)$
sign	$\mathrm{sign}(t)$		$\dfrac{2}{j\omega}$
rect	$\mathrm{rect}(at)$		$\dfrac{1}{\lvert a\rvert}\,\mathrm{sinc}\left(\dfrac{\omega}{2\pi a}\right)$
sinc	$\mathrm{sinc}(at)$		$\dfrac{1}{\lvert a\rvert}\,\mathrm{rect}\left(\dfrac{\omega}{2\pi a}\right)$
exponential	$e^{-at}1(t)$,	$a > 0$	$\dfrac{1}{j\omega + a}$
	$e^{at}1(-t)$,	$a > 0$	$\dfrac{1}{a - j\omega}$
	$e^{-a\lvert t\rvert}$,	$a > 0$	$\dfrac{2a}{\omega^2 + a^2}$
	$e^{-j\sigma t}$		$2\pi\,\delta(\omega + \sigma)$
sine	$\sin(\sigma t)$		$j\pi\left(\delta(\omega + \sigma) - \delta(\omega - \sigma)\right)$
cosine	$\cos(\sigma t)$		$\pi\left(\delta(\omega + \sigma) + \delta(\omega - \sigma)\right)$

	$x(t)$	$X(j\omega)$
linearity	$\alpha\,x(t) + \beta\,y(t)$	$\alpha\,X(j\omega) + \beta\,Y(j\omega)$
integration	$\displaystyle\int_{-\infty}^{t} x(\tau)\,d\tau$	$\dfrac{X(j\omega)}{j\omega} + \pi X(j0)\,\delta(\omega)$
differentiation	$\dot{x}(t) = \dfrac{d\,x(t)}{dt}$	$j\omega\,X(j\omega)$
	$x^{(n)}(t) = \dfrac{d^n\,x(t)}{dt^n}$	$(j\omega)^n\,X(j\omega)$
time product	$t\,x(t)$	$j\,\dfrac{d}{d\omega}X(j\omega)$
	$t^n x(t)$	$j^n\,\dfrac{d^n}{d\omega^n}X(j\omega)$
convolution	$x(t) * y(t)$	$X(j\omega)\,Y(j\omega)$
product	$x(t)\,y(t)$	$\dfrac{1}{2\pi}X(j\omega) * Y(j\omega)$
time shift	$x(t - \tau)$	$e^{-j\omega\tau}\,X(j\omega)$
frequency shift	$e^{-j\sigma t}x(t)$	$X(j(\omega + \sigma))$
time reversal	$x(-t)$	$X(-j\omega)$
time scaling	$x(a\,t)$	$\dfrac{1}{\lvert a\rvert}\,X\!\left(j\dfrac{\omega}{a}\right)$
frequency scaling	$\dfrac{1}{\lvert a\rvert}x\!\left(\dfrac{t}{a}\right)$	$X(ja\omega)$