MAE143A - Signals and Systems - Fall 2020
Midterm 2, November 19th

Instructions

1. You have until Thursday at 10:00PM to submit your answers on gradescope.
2. The exam is to be done individually.
3. All answers should be manuscript.
4. No typed answers of any form will be accepted.
5. All graphs and diagrams have to be sketched by hand. Include enough detail.
6. If you use a tablet to do the work make sure it is handwritten.
7. This exam has 4 questions, for a total of 40 points and 1 bonus points.

Notes on Academic integrity

1. Unlikely answers that look similar will be flagged as cheating. Your grade might be impacted and you may be reported.
2. We will complement the exam with an oral exam for a randomly selected number of students. It will work like this:
   - Next Wednesday, November 25th, during discussion session all students have to be present at the beginning of the session.
   - A number of students will be randomly selected for oral interview to take place during discussion session. It is mandatory that all students attend the beginning of the discussion. If your name is selected and you are not available for interview you will immediately receive a zero on the midterm. If you have any problems attending this session please contact me BEFORE next Wednesday so that we can work out an alternative.
   - In this in person (zoom) interview you will have to provide oral arguments to support the solutions that you turned in.
   - If you cannot successfully support your written answer your grade will be impacted and you might be reported for cheating.
3. For your information, with the numbers I am working on as of now, if 15 students cheat in the exam there is a 95% chance that at least one of them will be interviewed. If 20 students cheat, the chance is 98%, and if 25 cheat, 99.6%. You get the idea, right?

1. **Continuous-time System Response**

   A continuous-time linear time-invariant system has as impulse response

$$g(t) = \delta(t) - 10\, e^{-10\, t} 1(t).$$

   (a) (3 points) Calculate the system's transfer-function $G(s)$ and its region of convergence. *Hint: use any method you would like.*

Solution: Because the system is causal, we can use the Laplace transform.

$$\begin{aligned} G(s) &= \mathcal{L}\{g(t)\} \\ &= \mathcal{L}\{\delta(t)\} - \mathcal{L}\{e^{-10t}1(t)\} && \text{(+1 point)} \\ &= 1 - \frac{10}{s+10} = \frac{s+10-10}{s+10} = \frac{s}{s+10} && \text{(+1 point)} \end{aligned}$$

The region of convergence is $\mathrm{Re}\{s\} > -10$. (+1 point)

(b) (1 point) Write the ordinary differential equation corresponding to the system.

Solution: By inspection from the transfer-function

$$\dot{y} + 10\,y = \dot{u} \qquad \text{(+1 point)}$$

(c) (1 point) Is the system causal? Explain.

Solution: Yes because $g(t) = 0$, $t < 0$. (+1 point)

(d) (1 point) Is the system asymptotically stable? Explain.

Solution: Yes because the pole of $G(s)$ is $s = -10$ which has negative real part.
(+1 point)

(e) (4 points) Use the Fourier transform to calculate the output response of the system, $y(t)$, to the input

$$u(t) = \mathrm{rect}(t).$$

Sketch the response.

Solution: Calculate the transform of the input

$$U(j\omega) = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right) = \frac{\sin(\omega/2)}{\omega/2} = \frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega} \qquad \text{(+1 point)}$$

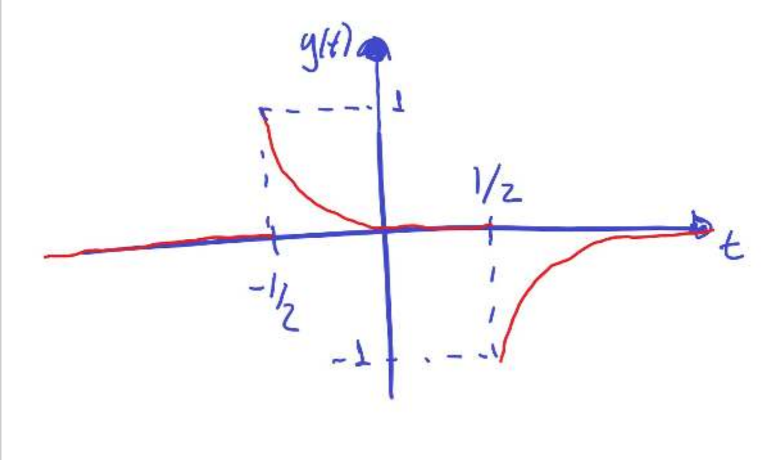
then the transform of the output

$$\begin{aligned} Y(j\omega) &= G(j\omega)\,U(j\omega) \\ &= \frac{j\omega}{j\omega+10}\,\frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega} \\ &= \frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega+10}. && \text{(+1 point)} \end{aligned}$$

The time-output can be obtained by the inverse Fourier transform

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\left\{\frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega+10}\right\} \\ &= \mathcal{F}^{-1}\left\{\frac{e^{j\omega/2}}{j\omega+10}\right\} - \mathcal{F}^{-1}\left\{\frac{e^{-j\omega/2}}{j\omega+10}\right\} \\ &= e^{-10(t+1/2)}1(t+1/2) - e^{-10(t-1/2)}1(t-1/2) && \text{(+1 point)} \end{aligned}$$

A sketch of the solution should look like:



(+1 point)

2. **Discrete-time System Response**

A discrete-time system is modeled by the difference equation

$$y[k+1] = a\,y[k] + (1-a)\,u[k], \quad 0 \le a < 1.$$

(a) (2 points) Calculate the system's transfer-function $G(z)$ and its region of convergence.

Solution: By inspection from the difference equation

$$G(z) = \frac{1-a}{z-a}.$$ (+1 point)

Its region of convergence is $|z| > a$. (+1 point)

(b) (1 point) Is the system causal? Explain.

Solution: Yes because $G(z)$ is proper. (+1 point)

(c) (1 point) Is the system asymptotically stable? Explain.

Solution: Yes because $|a| < 1$ and the pole $z = a$ is inside the unit circle. (+1 point)

(d) (3 points) Assume zero-initial conditions and calculate the response of the system to a unit step input. *Hint: It is OK to construct the solution iteratively.*

Solution: Iterate

$$\begin{aligned}
&y[0] = 0,\\
&y[1] = ay[0] + (1-a)u[0] = (1-a),\\
&y[2] = ay[1] + (1-a)u[1] = a(1-a) + (1-a) = (1+a)(1-a),\\
&y[3] = ay[2] + (1-a)u[2] = a(1+a)(1-a) + (1-a) = (1+a+a^2)(1-a),\\
&y[4] = ay[3] + (1-a)u[3] = a(1+a+a^2)(1-a) + (1-a) = (1+a+a^2+a^3)(1-a),\\
&\;\cdots\\
&y[k] = (1+a+a^2+\cdots+a^{k-1})(1-a) = (1-a)\sum_{\ell=0}^{k-1} a^\ell
\end{aligned}$$ (+3 points)

NOTE TO GRADERS: It is OK to end here.
One may also want to find an explicit formula

$$y[k] = (1-a)\frac{a^k - 1}{a-1} = (1-a^k)1[k].$$

(e) (2 points) Calculate the magnitude and phase of the system's frequency response $G(e^{j\omega})$.

Solution:

$$G(e^{j\omega}) = \frac{1-a}{e^{j\omega}-a} = \frac{1-a}{\cos(\omega)+j\sin(\omega)-a}$$

and

$$|G(e^{j\omega})| = \frac{|1-a|}{\sqrt{\sin(\omega)^2+(\cos(\omega)-a)^2}} = \frac{|1-a|}{\sqrt{1+a^2-2a\cos(\omega)}}, \qquad \text{(+1 point)}$$

$$\angle G(e^{j\omega}) = -\tan^{-1}\frac{\sin(\omega)}{\cos(\omega)-a} \qquad \text{(+1 point)}$$

(f) (1 point) Sketch the magnitude of the frequency response.

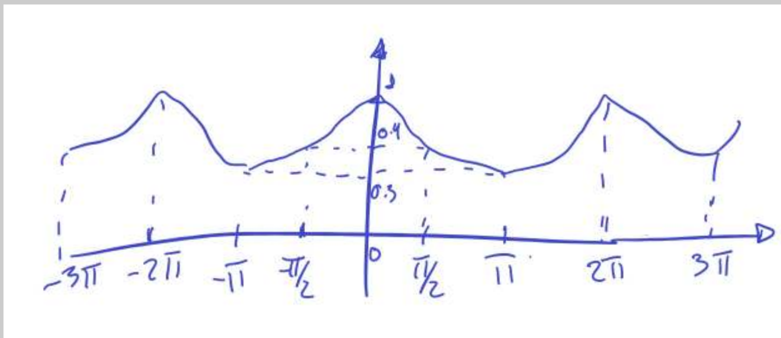
Solution:
Things to note: $|G(e^{j\omega})|$ is even and periodic. The period is 2π. A couple of points help with the magnitude in $[0,\pi]$:

$$\begin{aligned}
&\omega = 0 && |G(e^{j\omega})| = \frac{|1-a|}{\sqrt{1+a^2-2a}} = \frac{|1-a|}{\sqrt{(1-a)^2}} = 1\\
&\omega = \pi/2 && |G(e^{j\omega})| = \frac{|1-a|}{\sqrt{1+a^2}} = \frac{|1-a|}{\sqrt{1+a^2}}\\
&\omega = \pi && |G(e^{j\omega})| = \frac{|1-a|}{\sqrt{1+a^2+2a}} = \frac{1-a}{1+a} < 1
\end{aligned}$$

For example if $a = 1/2$

$$\begin{aligned}
&\omega = 0 && |G(e^{j\omega})| = 1\\
&\omega = \pi/2 && |G(e^{j\omega})| = \frac{1-a}{1+a^2} = \frac{1/2}{5/4} = \frac{2}{5} \approx 0.4\\
&\omega = \pi && |G(e^{j\omega})| = \frac{1-a}{1+a} = \frac{1/2}{3/2} = \frac{1}{3} \approx 0.3
\end{aligned}$$

Solution should look like:



(+1 point)

NOTE TO GRADERS: Things to consider: is it periodic? Is it even? The particular a chosen is not important.

3. **Frequency Response**

   Consider a continuous-time LTI system with transfer-function $G(s)$. Assume that $G(s)$ is asymptotically stable and with real coefficients.

   (a) (1 point) Is $G(j\omega)$ the Fourier transform of the impulse response? Explain.

   Solution: Yes because $G(s)$ is asymptotically stable and $G(s = j\omega) = \mathcal{F}\{g(t)\}$.
   (+1 point)

   (b) (2 points) Calculate the Fourier transform of the input signal $u(t) = \cos(\sigma t + \phi)$.

   Solution: Calculate

$$
\begin{aligned}
U(j\omega) &= \frac{1}{2}\mathcal{F}\{e^{j(\sigma t+\phi)} + e^{-j(\sigma t+\phi)}\} \\
&= \frac{e^{j\phi}}{2}\mathcal{F}\{e^{j\sigma t}\} + \frac{e^{-j\phi}}{2}\mathcal{F}\{e^{-j\sigma t}\} && \text{(+1 point)} \\
&= \pi e^{j\phi}\delta(\omega - \sigma) + \pi e^{-j\phi}\delta(\omega + \sigma) && \text{(+1 point)}
\end{aligned}
$$

   (c) (2 points) Calculate the Fourier transform, $Y(j\omega)$, of the output signal $y(t)$ if $u(t) = \cos(\sigma t + \phi)$ is the input.

   Solution: Calculate

$$
\begin{aligned}
Y(j\omega) &= G(j\omega)\,U(j\omega) \\
&= \pi e^{j\phi}G(j\omega)\delta(\omega - \sigma) + \pi e^{-j\phi}G(j\omega)\delta(\omega + \sigma) && \text{(+1 point)} \\
&= \pi e^{j\phi}G(j\sigma)\delta(\omega - \sigma) + \pi e^{-j\phi}G(-j\sigma)\delta(\omega + \sigma) && \text{(+1 point)}
\end{aligned}
$$

   (d) (4 points) Calculate the output response $y(t)$ by calculating the inverse Fourier transform of $Y(j\omega)$ from part (c).

Solution: Calculate

$$\begin{aligned}
y(t) &= \mathcal{F}\{Y(j\omega)\} \\
&= \frac{1}{2}e^{j\phi}G(j\sigma)\mathcal{F}\{2\pi\delta(\omega-\sigma)\} + \frac{1}{2}e^{-j\phi}G(-j\sigma)\mathcal{F}\{2\pi\delta(\omega+\sigma)\} && \text{(+1 point)} \\
&= \frac{1}{2}e^{j\phi}G(j\sigma)e^{j\sigma t} + \frac{1}{2}e^{-j\phi}G(-j\sigma)e^{-j\sigma t} && \text{(+1 point)} \\
&= \frac{1}{2}|G(j\sigma)|e^{j(\sigma t+\phi+\angle G(j\sigma))} + \frac{1}{2}|G(j\sigma)|e^{-j(\sigma t+\phi+\angle G(j\sigma))} && \text{(+1 point)} \\
&= |G(j\sigma)|\cos(\sigma t+\phi+\angle G(j\sigma)) && \text{(+1 point)}
\end{aligned}$$

(e) (1 point) Compare your answer from part (d) with the answer you would have obtained using the frequency response method.

Solution: The solutions obtained are identical. (+1 point)

4. **Amplifier**

In response to an input signal $u(t)$ an amplifier produces the output

$$y(t) = \begin{cases} 1, & u(t) \geq 1/2\ , \\ 2\,u(t), & |u(t)| < 1/2\ , \\ -1, & u(t) \leq -\ 1/2\ . \end{cases}$$

Consider a periodic input signal

$$u(t) = \cos(t)$$

and answer the following questions:

(a) (2 points) Determine the range of t in which $|u(t)| \geq 1/2$ and use that to write a piecewise description for the signal $y(t)$. Sketch the signals $u(t)$ and $y(t)$.

Solution:
First note that

$$\cos(t) \geq 1/2$$

whenever

$$|t - 2\kappa\pi| \leq T = \arccos(1/2) = \pi/3$$

and

$$\cos(t) \leq -1/2$$

whenever

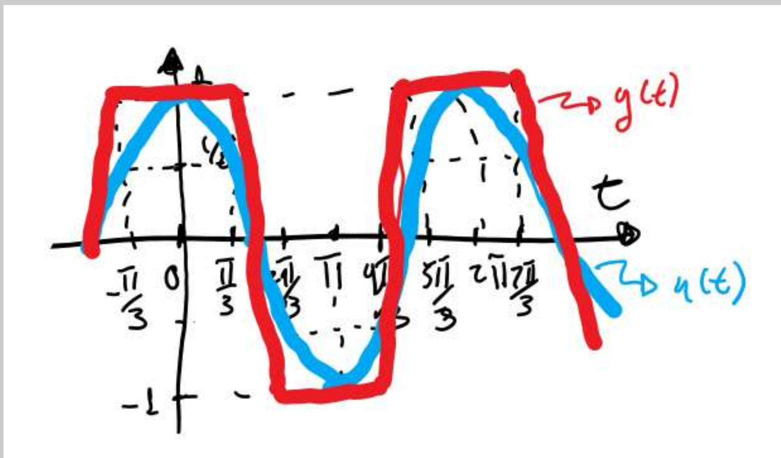
$$|t - 2(\kappa+1)\pi| \leq \pi/3$$

With that

$$y(t) = \begin{cases} 1, & |t - 2\kappa\pi| \leq \pi/3, \\ 2\cos(t), & |t - \pi/2 - \kappa\pi| \leq \pi/6, \\ -1, & |t - 2(\kappa+1)\pi| \leq \pi/3, \end{cases}$$

(+1 points)

Sketches should look like:



(+1 point)

NOTE TO GRADERS: Look for a good sketch with the key points.

(b) (1 point) Let

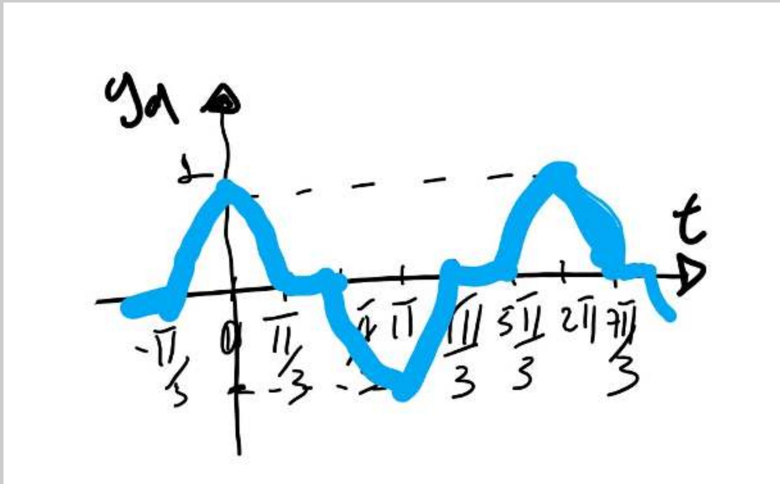
$$y_d(t) = 2\,u(t) - y(t)$$

be the *distortion signal*. Sketch $y_d(t)$.

Solution:

$$y_d = 2\,u(t) - y(t) = \begin{cases} 2\cos(t) - 1, & |t - 2\kappa\pi| \leq \pi/3, \\ 0, & |t - \pi/2 - \kappa\pi| \leq \pi/6, \\ 2\cos(t) + 1, & |t - 2(\kappa+1)\pi| \leq \pi/3, \end{cases}$$

Plots should look like:



(+1 point)

(c) (2 points) Show that the coefficients of the exponential Fourier series of $y_d(t)$ for $n = 0$ and $n = 1$ are

$$\hat{Y}_d[0] = 0, \qquad\qquad \hat{Y}_d[1] = \frac{2}{3} - \frac{\sqrt{3}}{2\pi}.$$

Solution: By inspection, the average of the signal y_d is zero, therefore $\hat{Y}_d[0] = 0$.

(+1 point)

For $n = 1$

$$\begin{aligned}
\hat{Y}_d[1] &= \frac{1}{2\pi}\int_{-\pi/3}^{\pi/3} (2\cos(t) - 1)\, e^{-jt}\, dt + \frac{1}{2\pi}\int_{\pi-\pi/3}^{\pi+\pi/3} (2\cos(t) + 1)\, e^{-jt}\, dt, \quad t = \pi + \tau \\
&= \frac{1}{2\pi}\int_{-\pi/3}^{\pi/3} (2\cos(t) - 1)\, e^{-jt}\, dt - \frac{e^{-j\pi}}{2\pi}\int_{-\pi/3}^{\pi/3} (2\cos(\tau) - 1)\, e^{-j\tau}\, d\tau, \\
&= \frac{1}{\pi}\int_{-\pi/3}^{\pi/3} (2\cos(t) - 1)\, e^{-jt}\, dt \\
&= \frac{1}{\pi}\int_{-\pi/3}^{\pi/3} (e^{jt} + e^{-jt})\, e^{-jt}\, dt - \frac{1}{\pi}\int_{-\pi/3}^{\pi/3} e^{-jt}\, dt \\
&= \frac{1}{\pi}\int_{-\pi/3}^{\pi/3} (1 - e^{-jt} + e^{-2jt})\, dt \\
&= \frac{1}{\pi}\left(1 + \frac{e^{-jt}}{j} - \frac{e^{-2jt}}{2j}\right)\Bigg|_{-\pi/3}^{\pi/3} \\
&= \frac{1}{\pi}\frac{2\pi}{3} + \frac{1}{\pi}\frac{e^{2j\pi/3} - e^{-2j\pi/3}}{2j} - \frac{1}{\pi}\frac{e^{j\pi/3} - e^{-j\pi/3}}{j} \\
&= \frac{2}{3}\frac{1}{\pi}(\sin(2\pi/3) - 2\sin(\pi/3)) = \frac{2}{3} - \frac{\sqrt{3}}{2\pi}
\end{aligned}$$

(+1 point)

(d) (1 point (bonus)) Calculate all other coefficients of the exponential Fourier series of $y_d(t)$.

Solution:
For those who have the stomach the solution is:

$$\hat{Y}_d[n] = \begin{cases} \frac{2}{\pi} \frac{n\sqrt{3}\cos(n\pi/3) - \sin(n\pi/3)}{n(1-n^2)}, & n \text{ is odd,} \\ 0 & n \text{ is even.} \end{cases}$$

(+1 bonus point)

(e) (1 point) Without calculating any integrals, show how the coefficients of the exponential Fourier series of $y_d(t)$ can be used to calculate the coefficients of the Fourier series of $y(t)$. *Hint: You do not need to have calculated the coefficients to answer this question!*

Solution: Because

$$\begin{aligned} y(t) &= 2u(t) - y_d(t) \\ &= 2\cos(\omega_0 t) - \sum_{n=-\infty}^{\infty} \hat{Y}_d[n] e^{jn\omega_o t} \\ &= e^{j\omega_0 t} + e^{-j\omega_0 t} - \sum_{n=-\infty}^{\infty} \hat{Y}_d[n] e^{jn\omega_o t} \end{aligned}$$

so that

$$y(t) = \sum_{n=-\infty}^{\infty} \hat{Y}[n] e^{jn\omega_o t}$$

in which

$$\hat{Y}[n] = \begin{cases} -\hat{Y}_d[n], & n \neq \pm 1, \\ 1 - \hat{Y}_d[1], & n = \pm 1. \end{cases}$$

(+1 point)

(f) (2 points) The average power of a periodic signal is

$$P_x = \frac{1}{T_0} \int_0^{T_0} x(t)^2 \, dt.$$

Calculate the average power of $u(t)$ and $y_d(t)$.
Hint:

$$\int \cos(t)^2 \, dt = \frac{t}{2} + \frac{1}{4}\sin(2t).$$

Solution:
Taking advantage of the symmetry

$$\begin{aligned} P_u &= \frac{1}{\pi} \int_0^{\pi} \cos(t)^2 \, dt \\ &= \frac{1}{\pi} \left(\frac{t}{2} + \frac{1}{4}\sin(2t) \Big|_0^{\pi} \right) \\ &= \frac{1}{\pi} \left(\frac{\pi}{2} \right) = \frac{1}{2} \end{aligned}$$

(+1 point)

and

$$\begin{aligned}
P_{y_d} &= \frac{2}{\pi}\int_0^{\pi/3} (2\cos(t) - 1)^2\, dt \\
&= \frac{2}{\pi}\int_0^{\pi/3} (4\cos(t)^2 - 4\cos(t) + 1)\, dt \\
&= \frac{2}{\pi}\left(\frac{2\pi}{3} + \sin(2\pi/3) - 4\sin(\pi/3) + \frac{\pi}{3}\right) \\
&= \frac{2}{\pi}\left(\pi - \frac{3\sqrt{3}}{2}\right) \\
&= 2 - \frac{3\sqrt{3}}{\pi} \approx 0.35 \qquad \text{(+1 point)}
\end{aligned}$$

NOTE TO GRADERS: Stop here.
Just for completeness

$$\begin{aligned}
P_y &= \frac{1}{\pi}\left(\int_0^{\pi/3} dt + \int_{\pi/3}^{2\pi/3} 4\cos(t)^2\, dt + \int_{2\pi/3}^{\pi} dt\right) \\
&= \frac{1}{\pi}\left(\frac{\pi}{3} + \frac{4\pi}{3} - \frac{2\pi}{3} + \sin(4\pi/3) - \sin(2\pi/3) + \frac{\pi}{3}\right) \\
&= \frac{1}{\pi}\left(\frac{4\pi}{3} + (\sin(4\pi/3) - \sin(2\pi/3))\right) \\
&= \frac{1}{\pi}\left(\frac{4\pi}{3} - \sqrt{3}\right) \\
&= \frac{4}{3} - \frac{\sqrt{3}}{\pi} \approx 0.78
\end{aligned}$$

(g) (2 points) Use question (e) and Parseval's theorem to show that

$$P_y = P_{y_d} + 2(1 - 2\,\mathrm{Re}\{\hat{Y}_d[1]\}),$$

in which $\hat{Y}_d[1]$ is the coefficient $n = 1$ from item (c). Calculate P_y using parts (c) and (f).

Solution: Use Parseval's theorem to rewrite

$$\begin{aligned}
P_y &= \sum_{n=-\infty}^{\infty} |\hat{Y}[n]|^2 \\
&= \sum_{n=-\infty, n\neq\pm1}^{\infty} |\hat{Y}_d[n]|^2 + 2|1 - \hat{Y}_d[1]|^2 \\
&= 2(1 - \hat{Y}_d[1])(1 - \overline{\hat{Y}_d[1]}) - 2|\hat{Y}_d[1]|^2 + \sum_{n=-\infty}^{\infty} |\hat{Y}_d[n]|^2 \\
&= 2(1 - \hat{Y}_d[1] - \overline{\hat{Y}_d[1]}) + P_{y_d}. \qquad \text{(+1 point)}
\end{aligned}$$

Pluggin in the formula

$$
\begin{aligned}
P_{y_d} + 2(1 - \hat{Y}_d[1] - \overline{\hat{Y}_d[1]}) &= 2 - \frac{3\sqrt{3}}{\pi} + 2\left(1 - 2\left(\frac{2}{3} - \frac{\sqrt{3}}{2\pi}\right)\right) \\
&= 2 - \frac{3\sqrt{3}}{\pi} + 2 - \frac{8}{3} - \frac{2\sqrt{3}}{\pi} \\
&= \frac{4}{3} - \frac{\sqrt{3}}{\pi} \qquad \text{(+1 point)}
\end{aligned}
$$

NOTE TO GRADERS: Students get marks for evaluation even if they did not prove the formula.

	$x(t)$		$X(j\omega)$
impulse	$\delta(t)$		1
	$\delta^{(n)}(t)$		$(j\omega)^n$
constant	1		$2\pi\,\delta(\omega)$
step	$1(t)$		$\dfrac{1}{j\omega} + \pi\,\delta(\omega)$
sign	$\mathrm{sign}(t)$		$\dfrac{2}{j\omega}$
rect	$\mathrm{rect}(at)$		$\dfrac{1}{\lvert a\rvert}\,\mathrm{sinc}\left(\dfrac{\omega}{2\pi a}\right)$
sinc	$\mathrm{sinc}(at)$		$\dfrac{1}{\lvert a\rvert}\,\mathrm{rect}\left(\dfrac{\omega}{2\pi a}\right)$
exponential	$e^{-at}1(t)$,	$a > 0$	$\dfrac{1}{j\omega + a}$
	$e^{at}1(-t)$,	$a > 0$	$\dfrac{1}{a - j\omega}$
	$e^{-a\lvert t\rvert}$,	$a > 0$	$\dfrac{2a}{\omega^2 + a^2}$
	$e^{-j\sigma t}$		$2\pi\,\delta(\omega + \sigma)$
sine	$\sin(\sigma t)$		$j\pi\left(\delta(\omega+\sigma) - \delta(\omega-\sigma)\right)$
cosine	$\cos(\sigma t)$		$\pi\left(\delta(\omega+\sigma) + \delta(\omega-\sigma)\right)$

	$x(t)$	$X(j\omega)$
linearity	$\alpha\, x(t) + \beta\, y(t)$	$\alpha\, X(j\omega) + \beta\, Y(j\omega)$
integration	$\displaystyle\int_{-\infty}^{t} x(\tau)\, d\tau$	$\dfrac{X(j\omega)}{j\omega} + \pi X(j0)\,\delta(\omega)$
differentiation	$\dot{x}(t) = \dfrac{d\,x(t)}{dt}$	$j\omega\, X(j\omega)$
	$x^{(n)}(t) = \dfrac{d^n\,x(t)}{dt^n}$	$(j\omega)^n\, X(j\omega)$
time product	$t\,x(t)$	$j\,\dfrac{d}{d\omega}X(j\omega)$
	$t^n x(t)$	$j^n\,\dfrac{d^n}{d\omega^n}X(j\omega)$
convolution	$x(t) * y(t)$	$X(j\omega)\,Y(j\omega)$
product	$x(t)\,y(t)$	$\dfrac{1}{2\pi}X(j\omega) * Y(j\omega)$
time shift	$x(t-\tau)$	$e^{-j\omega\tau}\,X(j\omega)$
frequency shift	$e^{-j\sigma t}x(t)$	$X(j(\omega+\sigma))$
time reversal	$x(-t)$	$X(-j\omega)$
time scaling	$x(a\,t)$	$\dfrac{1}{\lvert a\rvert}\,X\!\left(j\dfrac{\omega}{a}\right)$
frequency scaling	$\dfrac{1}{\lvert a\rvert}x\!\left(\dfrac{t}{a}\right)$	$X(ja\omega)$