MAE143 A - Signals and Systems - Winter 11
Final, March 15th

Instructions

(i) This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities

(ii) You have 90 minutes

(iii) Do not forget to write your name, student number, and instructor

1. **Continuous-time Fourier Series and Fourier Transform**

   A periodic signal $x(t)$ with period $T = 3$ s has a harmonic series

$$X[0] = 1, \qquad X[1] = X[-1]^* = \frac{1}{2}, \qquad X[2] = X[-2]^* = 0, \qquad X[3] = X[-3]^* = \frac{j}{2},$$

   and $X[k] = 0$ for all $|k| \geq 4$. Answer the following questions regarding this signal:

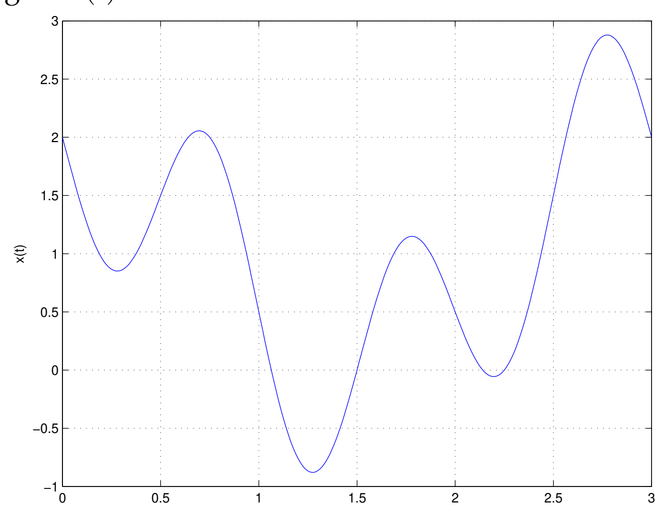
   (a) (2 points) Compute the signal $x(t)$.

   Solution: The signal $x(t)$ is computed from its harmonic series

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} X[k]\, e^{j2\pi kt/T}, \quad T = 3 & \text{(+ 1 point)} \\ &= 1 + \frac{1}{2}e^{j2\pi t/3} + \frac{1}{2}e^{-j2\pi t/3} + \frac{j}{2}e^{j2\pi t} - \frac{j}{2}e^{-j2\pi t} \\ &= 1 + \cos 2\pi t/3 - \sin 2\pi t. & \text{(+ 1 point)} \end{aligned}$$

   (b) (2 points) Sketch the signal $x(t)$ over one period.

   Solution: The signal $x(t)$ looks like: (+ 2 points)



> **GRADERS:** Look for the value of the function at $t = 0$, $t = 1.5$ and $t = 3$. At all other points the plot does need to be exact!

(c) (2 points) Compute the Continuous-time Fourier Transform $X(f)$ of $x(t)$. Sketch the magnitude and phase of $X(f)$ versus f.

> **Solution:** The Continuous-time Fourier Transform of $x(t)$ is computed directly from its harmonic series
>
> $$X(f) = \delta(f) + \frac{1}{2}\delta(f + 1/3) + \frac{1}{2}\delta(f - 1/3) - \frac{j}{2}\delta(f + 1) + \frac{j}{2}\delta(f - 1) \qquad \text{(+ 1 point)}$$
>
> The spectrum $X(f)$ looks like: (+ 1 point)
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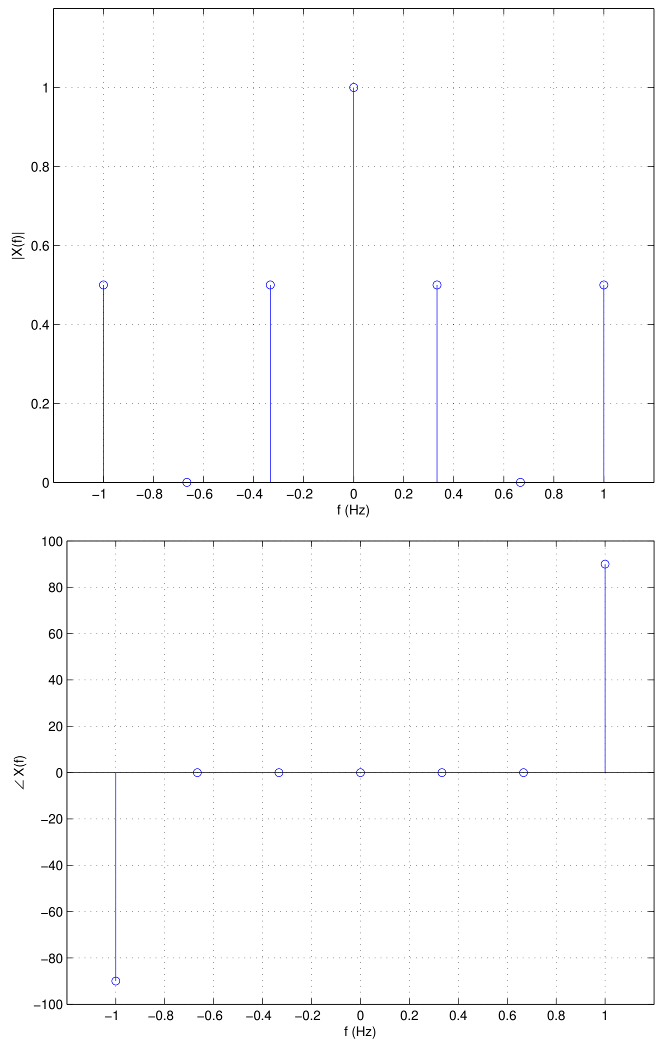
> 
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Total for Question 1: 6

2. **Filtering**

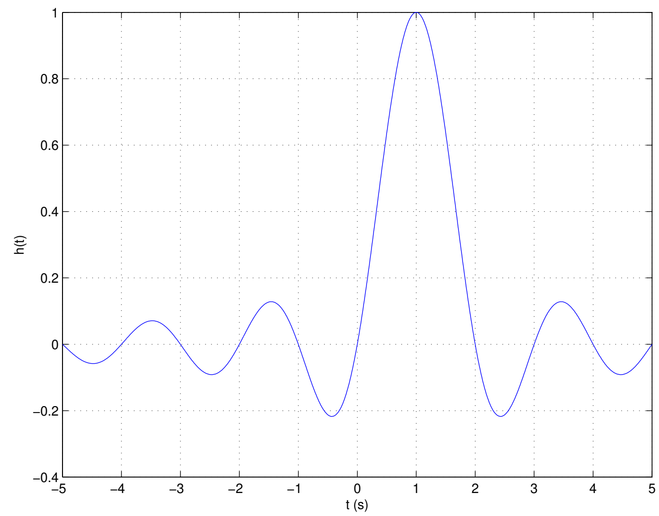
A linear time-invariant system has as impulse response

$$h(t) = \mathrm{sinc}(t-1).$$

Answer the following questions regarding this system:

(a) (1 point) Sketch the impulse response $h(t)$.

Solution: The impulse response $h(t)$ looks like: (+ 1 point)

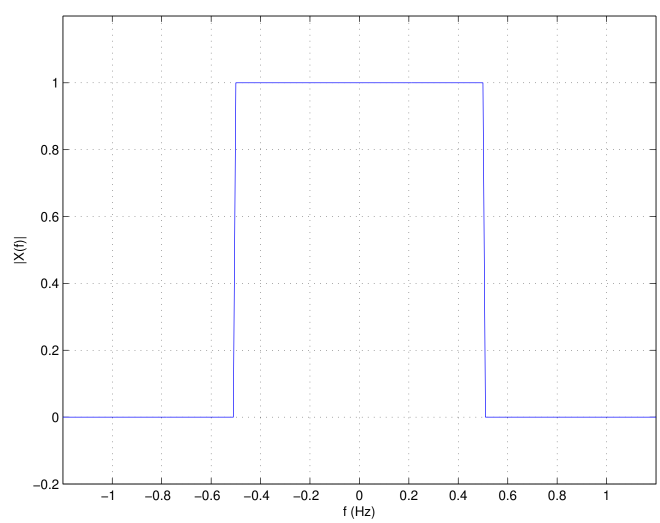


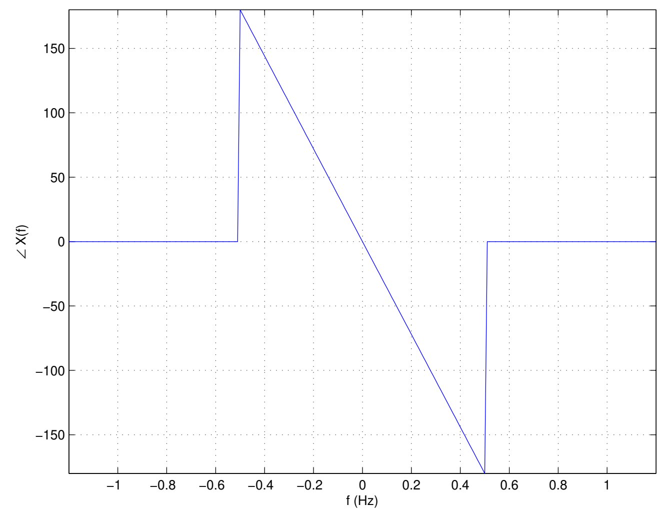
(b) (2 points) Compute its frequency response $H(f)$. Sketch the magnitude and phase of $H(f)$ versus f.

Solution: The frequency response $H(f)$ is obtained by taking the CTFT of $h(t)$

$$H(f) = \mathcal{F}\{h(t)\} = \mathcal{F}\{\mathrm{sinc}(t-1)\} = \mathcal{F}\{\mathrm{sinc}(t)\}e^{-j2\pi f} = \mathrm{rect}(f)\, e^{-j2\pi f}. \quad \text{(+ 1 point)}$$

The frequency response $H(f)$ looks like: (+ 1 point)





(c) (2 points) What type of filter is this system? What is the filter bandwidth? Is this filter ideal? Is this filter causal? Explain your answer.

Solution:
This is a low pass filter. (+ 1/2 point)
The bandwidth is $1/2$ Hz. (+ 1/2 point)
It is an ideal filter because it does not distort the input signal: magnitude is constant over the passband and phase is a linear function of f. (+ 1/2 point)
The filter is non-causal because the impulse response assume non-zero values for $t < 0$. (+ 1/2 point)

(d) (2 points) Show that the steady-state response of this system to the input signal $x(t)$ described in Question 1 is

$$y_{ss}(t) = 1 + \cos(2\pi(t-1)/3).$$

Solution:
The steady-state solution $y_{ss}(t)$ can be computed using the frequency response. In this case $y_{ss}(t) = \mathcal{F}^{-1}\{Y_{ss}(f)\}$, where

$$Y_{ss}(f) = H(f)X(f) = \delta(f) + \frac{e^{j2\pi/3}}{2}\delta(f+1/3) + \frac{e^{-j2\pi/3}}{2}\delta(f-1/3) \qquad \text{(+ 1 point)}$$

because the system is a low-pass filter with cut-off frequency $1/2$ Hz.
Hence

$$y_{ss}(t) = \mathcal{F}^{-1}\{Y_{ss}(f)\} = 1 + \cos(2\pi t/3 - 2\pi/3) \qquad \text{(+ 1 point)}$$

which matches the signal given in the problem statement.

Total for Question 2: 7

3. **Sampling and Signal Reconstruction**

   Answer the following questions regarding the signal $x(t)$ described in Question 1 and the filter described in Question 2.

   (a) (1 point) What is the bandwidth of $x(t)$?

   Solution: The signal $x(t)$ is band-limited and its bandwidth is equal to its highest harmonic which is $f_B = 1$ Hz. (+1 point)

   (b) (1 point) What is the range of sampling frequencies f_s with which you could sample the signal $x(t)$ and later recover it exactly using an ideal low-pass filter?

   Solution: From the sampling theorem, we need to have $f_s > 2f_B = 2$ Hz. (+1 point)

   (c) (2 points) Sketch the magnitude and phase of the spectrum $X_3(f)$ of the signal $x(t)$ after it has been sampled at frequency $f_s = 3$ Hz.

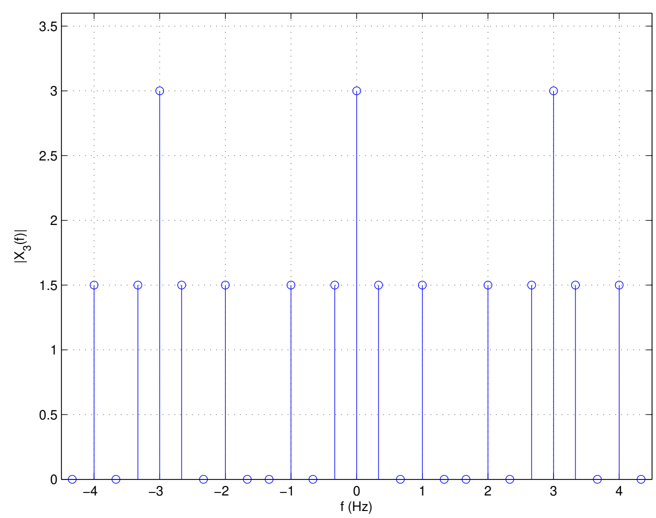
   Solution: The spectrum of a sampled signal is the periodic function

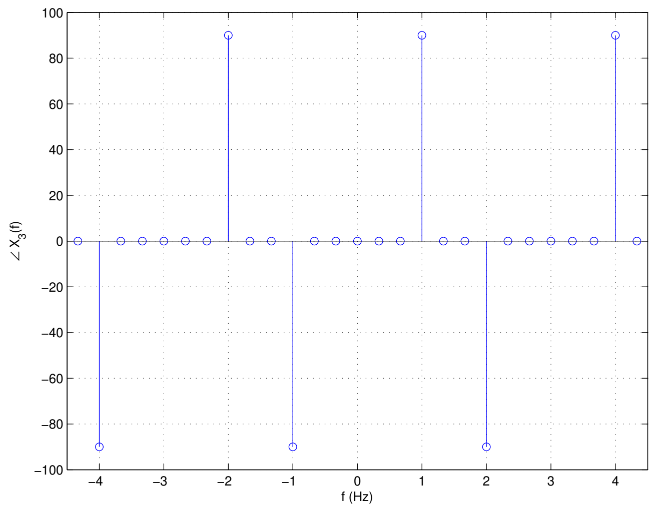
$$X_{f_s}(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

   For $f_s = 3$ we have

$$X_3(f) = 3 \sum_{n=-\infty}^{\infty} X(f - 3n) \qquad \text{(+1 point)}$$

   The sampled spectrum is then obtained by scaling and repeating the answer obtained in Question 1 (c). (+1 point)





GRADERS: Please consider the answer correct also if it corresponde to correctly sampling an spectrum calculated incorrectly in Question 1.

(d) (2 points) Determine the signal $y_3(t)$ which is obtained after running the sampled signal you computed in part (c) through the filter in Question 2 until you reach steady state. Hint: You do not need any new computations to answer this question!

Solution:
The spectrum of $X_3(f)$ is the same as the spectrum of $X(f)$ in the bandpass of the filter in Question 2, hence the answer will be the same as in Question 2 (c) after scaling, that is

$$y_3(t) = 3y_{ss}(t) = 3 + 3\cos(2\pi(t-1)/3). \qquad \text{(+2 points)}$$

GRADERS: Of course one can also recompute $y_3(t)$ from its spectrum without noticing that it produces the same signal $y_{ss}(t)$.

(e) (1 point (bonus)) Sketch the magnitude and phase of the spectrum $X_2(f)$ of the signal $x(t)$ after it has been sampled at frequency $f_s = 2$ Hz.

Solution: As in part (c), the spectrum of a sampled signal is the periodic function

$$X_{f_s}(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

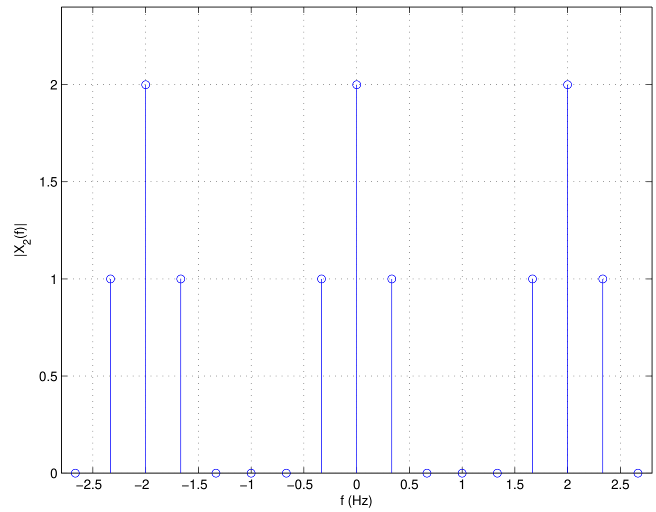
For $f_s = 1$ we have

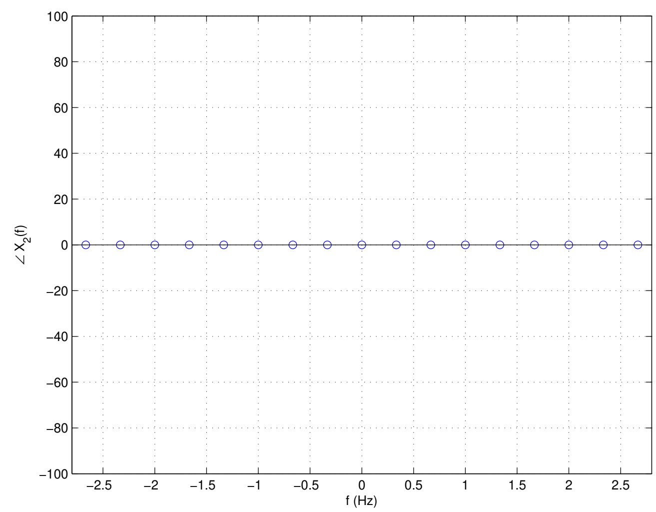
$$X_2(f) = 2 \sum_{n=-\infty}^{\infty} X(f - 2n)$$

Here we have to be careful with the harmonic at $f = n$. For example at $f = 1$ we have to add components from $X(f)$ and $X(f - 2)$ that is

$$X_2(1) = \frac{1}{2}\left(X(1) + X(-1)\right) = 0$$

That is, due to aliasing, the harmonic at $f = 1$ simply disappears from the spectrum! The resulting sampled spectrum is then obtained by scaling and repeating the answer to Question 1 (c) after suppressing the harmonic at $f = 1$ and $f = -1$. (+1 point)





(f) (1 point (bonus)) Determine the signal $y_2(t)$ which is obtained after running the sampled signal you computed in part (e) through the filter in Question 2 until you reach steady state.

Solution:
After comparing the spectrum of $X_2(f)$ with the spectrum of $X(f)$ in the bandpass of the filter in Question 2, we conclude that, once again, the answer will be the same as

in Question 2 (c) after scaling, that is

$$y_2(t) = 2y_{ss}(t) = 2 + 2\cos(2\pi(t-1)/3). \qquad \text{(+1 point)}$$

Total for Question 3: 6

Total for Question 3: 2 (bonus)