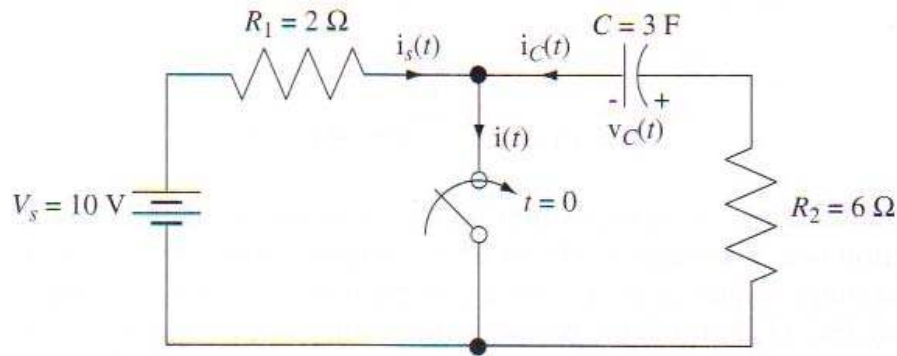


2.



$$\textcircled{1} \quad i(t) = i_s(t) + i_c(t) \quad \textcircled{2} \quad i_s(t) = \frac{V_s}{R_1}$$

$$\textcircled{3} \quad i_c(t) = C \frac{d}{dt}(V_c(t)) \quad \textcircled{4} \quad V_c(t) + i_c(t)R_2 = 0$$

$$\textcircled{3}, \textcircled{4} \Rightarrow \textcircled{5} \quad V_c(t) + C \frac{d}{dt}(V_c(t))R_2 = 0$$

The eigenvalue of the solution is $\lambda = -\frac{1}{CR_2}$, so the solution is $Ke^{\lambda t} = Ke^{-\frac{t}{CR_2}}$,

where K is a constant we will get from initial condition.

Initial condition: The power switch is open before $t = 0$, so it's just a simple series circuit containing V_s , R_1 , C and R_2 . The current of this series circuit is 0 and the capacitor store charges coming from V_s . Positive charges go to the left side of the capacitor while negative charges go to its right. $\Rightarrow V_c(0^+) = V_c(0^-) = -V_s = -10V$

$$\begin{cases} V_c(t) = Ke^{-\frac{t}{CR_2}} = Ke^{-\frac{t}{18}} \\ V_c(0) = -10V \end{cases} \Rightarrow K = -10, \quad V_c(t) = -10e^{-\frac{t}{18}}$$

$$\textcircled{2} \Rightarrow i_s(t) = 5$$

$$\textcircled{3} \Rightarrow i_c(t) = C \frac{d}{dt}(V_c(t)) = 3 \frac{d}{dt}(V_c(t)) = \frac{5}{3}e^{-\frac{t}{18}}$$

$$\textcircled{1} \Rightarrow i(t) = i_s(t) + i_c(t) = 5 + \frac{5}{3}e^{-\frac{t}{18}}$$

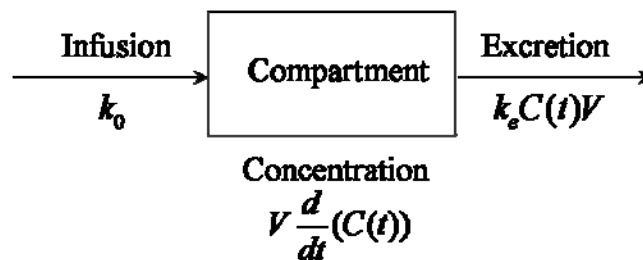
13. (a) V : the volume with unit l ;

$C(t)$: the total amount of drug concentration, with unit mg/l ;

k_0 : the amount of drug enters the compartment per hour, with unit mg/hr ;

k_e : the percentage of total concentrated drug that runs out of the compartment per hour, with unit hr^{-1} .

At some time point, drug enters the compartment is k_0 ; While total concentration amount is $C(t)V$, concentration created at this time point is $\frac{d}{dt}(C(t)V)$, with unit mg/hr ; Excretion is the percentage per hour k_e multiply by the total concentrated amount $C(t)V$.



Then,

Concentration = Infusion - Excretion

$$V \frac{d}{dt}(C(t)) = k_0 - k_e C(t)V$$

(b) Substitute with parameters $\Rightarrow 20 \frac{d}{dt}(C(t)) + 8C(t) = 200$

① Homogeneous solution: $20 \frac{d}{dt}(C(t)) + 8C(t) = 0$

The eigenvalue of the solution is $\lambda = -\frac{8}{20} = -0.4$, so the homogeneous

solution should be in the form of $C_h(t) = K_1 e^{-0.4t}$, where K_1 is a constant.

② Particular solution: As the right side of the equation is 200, the particular

solution should in a form of a constant K_2 .

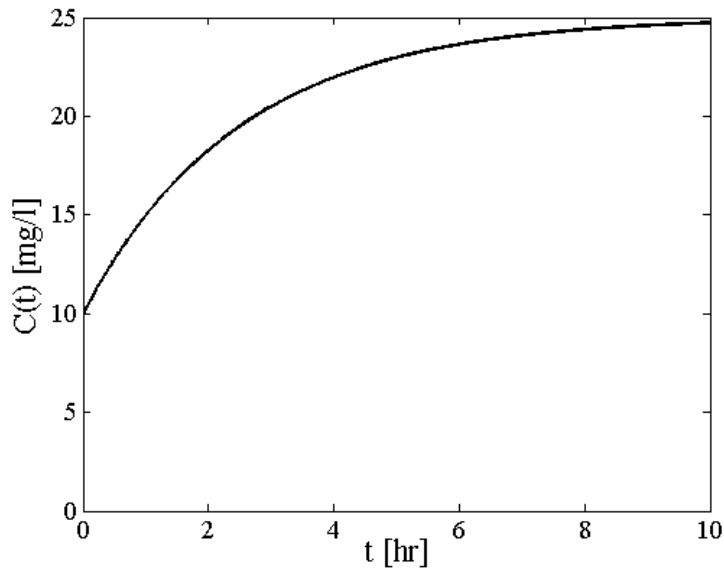
Put into the equation $\Rightarrow C_p(t) = K_2 = \frac{200}{8} = 25$

③ Final solution: $C(t) = C_h(t) + C_p(t) = K_1 e^{-0.4t} + 25$

Initial conditions: $C(0) = 10\text{mg/l}$

$\Rightarrow K_1 = -15$

$C(t) = (-15e^{-0.4t} + 25)\text{mg/l}$



16. (a) $f(t) - mg \sin(\theta) - k_f v(t) = ma(t),$

where $v(t) = \frac{d}{dt}(y(t)), a(t) = \frac{d^2}{dt^2}(y(t))$

$\Rightarrow m \frac{d^2}{dt^2}(y(t)) + k_f \frac{d}{dt}(y(t)) + mg \sin(\theta) = f(t)$

(b) No excitation force applied $\Rightarrow f(t) = 0$

$m \frac{d^2}{dt^2}(y(t)) + k_f \frac{d}{dt}(y(t)) = -mg \sin(\theta)$

① Homogeneous solution:

The eigenvalues of the solution are: $\lambda_1 = 0$, $\lambda_2 = -\frac{k_f}{m}$

So the homogeneous solution should in the form of:

$$y_h(t) = K_{h1}e^{\lambda_1 t} + K_{h2}e^{\lambda_2 t} = K_{h1} + K_{h2}e^{-\frac{k_f}{m}t}$$

② Particular solution:

The right side of the equation is a constant $-mg \sin(\theta)$ and the left side contains second order and first order derivatives, so the particular solution should in the form of a linear function of t , set as $y_p = K_p t$. Put into the

$$\text{equation} \Rightarrow K_p = -\frac{mg}{k_f} \sin(\theta), \quad y_p(t) = -\frac{mg}{k_f} \sin(\theta)t$$

③ Summation of homogeneous solution and particular solution:

$$y(t) = y_h(t) + y_p(t) = K_{h1} + K_{h2}e^{-\frac{k_f}{m}t} - \frac{mg}{k_f} \sin(\theta)t$$

Initial conditions:

$$y(0) = 0 = K_{h1} + K_{h2},$$

$$y'(0) = 10 = -\frac{k_f}{m} K_{h2} - \frac{mg}{k_f} \sin(\theta)$$

Substituting for the constants given in the problem statement: $m = 1000\text{kg}$,

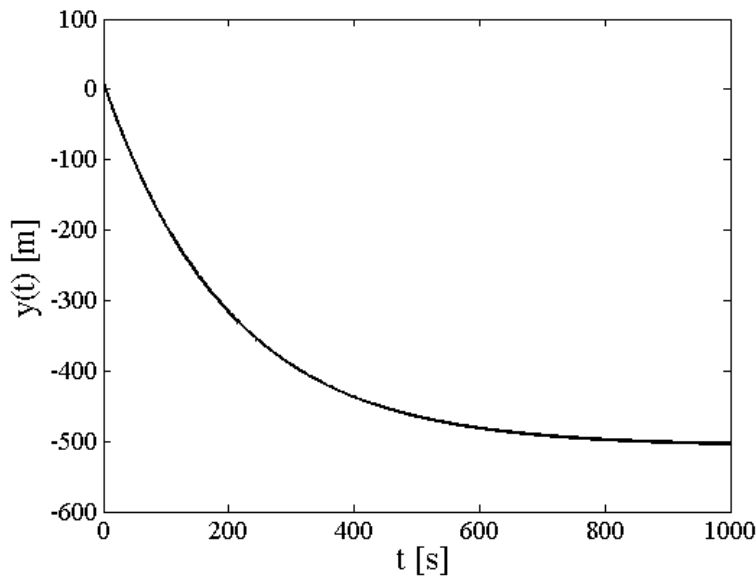
$$k_f = 5\text{N} \cdot \text{s/m}, \quad \theta = \pi / 12 \quad \text{and using } g = 9.8\text{m/s}^2$$

$$\Rightarrow K_{h2} = -\left(\frac{m}{k_f}\right)^2 g \sin(\theta) - 10 \frac{m}{k_f} = -1.0346 \times 10^5$$

$$K_{h1} = -K_{h2} = 1.0346 \times 10^5$$

$$y(t) = 1.0346 \times 10^5 (1 - e^{-\frac{t}{200}}) - 507.28t$$

$$y'(t) = 517.28e^{-\frac{t}{200}} - 507.28$$



(c) Constant force $f(t) = F = 200\text{N}$ is applied,

$$m \frac{d^2}{dt^2}(y(t)) + k_f \frac{d}{dt} y(t) = F - mg \sin(\theta)$$

① Homogeneous solution: same as part (b) ①

$$y_h(t) = K_{h1}e^{x_1 t} + K_{h2}e^{x_2 t} = K_{h1} + K_{h2}e^{-\frac{k_f}{m}t}$$

② Particular solution: similar to part (b) ②

The right side of the equation is a constant $F - mg \sin(\theta)$. Set particular solution as linear function of t , as $y_p = K_p t$. Put into the equation

$$\Rightarrow K_p = \frac{1}{k_f}[F - mg \sin(\theta)],$$

$$y_p(t) = \frac{t}{k_f}[F - mg \sin(\theta)]$$

③ Summation:

$$y(t) = y_h(t) + y_p(t) = K_{h1} + K_{h2}e^{-\frac{k_f}{m}t} + \frac{t}{k_f}[F - mg \sin(\theta)]$$

$$\frac{d}{dt}(y(t)) = \left(-\frac{k_f}{m}\right)K_{h2}e^{-\frac{k_f}{m}t} + \frac{1}{k_f}[F - mg \sin(\theta)]$$

$$\lim_{t \rightarrow \infty} \left(\frac{d}{dt}(y(t))\right) = \frac{1}{k_f}[F - mg \sin(\theta)] = -467.3 \text{ m/s}$$

A negative terminal velocity indicates the force of 200N is insufficient to move the car forward on that inclined plane, so the car is rolling backward down the hill.

17. System equation: $4y''(t) + y'(t) + 3y(t) = x(t)$

Linearity

$$x_1(t) = g(t) \Rightarrow 4y_1''(t) + y_1'(t) + 3y_1(t) = x_1(t) = g(t) \quad (1)$$

$$x_2(t) = h(t) \Rightarrow 4y_2''(t) + y_2'(t) + 3y_2(t) = x_2(t) = h(t) \quad (2)$$

$$x_3(t) = Kg(t) + h(t) \Rightarrow 4y_3''(t) + y_3'(t) + 3y_3(t) = x_3(t) = Kg(t) + h(t) \quad (3)$$

$$(1) \times K + (2) \Rightarrow$$

$$4Ky_1''(t) + 4y_2''(t) + Ky_1'(t) + y_2'(t) + 3Ky_1(t) + 3y_2(t) = Kg(t) + h(t) \quad (4)$$

Comparing (3) and (4) $\Rightarrow y_3(t) = Ky_1(t) + y_2(t)$

So this system is **linear**.

Time Invariance

$$x_1(t) = g(t) \Rightarrow 4y_1''(t) + y_1'(t) + 3y_1(t) = x_1(t) = g(t) \quad (1)$$

$$x_2(t) = g(t - t_0) \Rightarrow 4y_2''(t) + y_2'(t) + 3y_2(t) = x_2(t) = g(t - t_0) \quad (2)$$

Substituting t with $t - t_0$ in (1)

$$\Rightarrow 4y_1''(t - t_0) + y_1'(t - t_0) + 3y_1(t - t_0) = g(t - t_0) \quad (3)$$

Comparing (2) and (3)

$$\Rightarrow y_2(t) = y_1(t - t_0)$$

So this system is **time invariant**.

Stability

The eigenvalues of the solution of this system are:

$$\lambda_1 = -0.125 + j0.484$$

$$\lambda_2 = -0.125 - j0.484$$

Real parts of all eigenvalues are negative, so the system is **stable**.

Causality and Memory

The response at any time $t = t_0$ depends on the excitation at times $t < t_0$, not on any future values, so this system is **causal and has memory**.

Invertibility

The excitation of this system is expressed in terms of the response and its derivatives, so the excitation is uniquely determined by the response. This system is **invertible**.

18. System equation: $y(t) = x^3(t)$

Linearity

$$x_1(t) = g(t) \Rightarrow y_1(t) = x_1^3(t) = g^3(t) \quad \textcircled{1}$$

$$x_2(t) = h(t) \Rightarrow y_2(t) = x_2^3(t) = h^3(t) \quad \textcircled{2}$$

$$x_3(t) = Kg(t) + h(t) \Rightarrow y_3(t) = x_3^3(t) = [Kg(t) + h(t)]^3 \quad \textcircled{3}$$

$$\textcircled{1} \times K + \textcircled{2} \Rightarrow Ky_1(t) + y_2(t) = Kg^3(t) + h^3(t) \quad \textcircled{4}$$

$$\text{Comparing } \textcircled{3} \text{ and } \textcircled{4} \Rightarrow y_3(t) \neq Ky_1(t) + y_2(t)$$

So this system is **non-linear**.

Time Invariance

$$x_1(t) = g(t) \Rightarrow y_1(t) = x_1^3(t) = g^3(t) \quad \textcircled{1}$$

$$x_2(t) = g(t - t_0) \Rightarrow y_2(t) = x_2^3(t) = g^3(t - t_0) \quad \textcircled{2}$$

Substituting t with $t - t_0$ in $\textcircled{1}$

$$\Rightarrow y_1(t-t_0) = x_1^3(t-t_0) = g^3(t-t_0) \quad (3)$$

Comparing (2) and (3) $\Rightarrow y_2(t) = y_1(t-t_0)$

So this system is **time invariant**.

Stability

If $x(t)$ is bounded then its response $y(t)$ is bounded, so this system is **stable**.

Causality and Memory

The response at any time $t = t_0$ depends only on the excitation at time $t = t_0$, not on any future value or any past values, so this system is **causal and memoryless**.

Invertibility

The excitation can be written in terms of its response as: $x(t) = y^{\frac{1}{3}}(t)$

Two cases need to be considered:

① When excitation $x(t)$ is real, the cube root produces a unique excitation for a given response, so the system is **invertible for real $x(t)$** .

② When excitation $x(t)$ is complex, the cube root operation is multiple valued, so the excitation cannot be uniquely determined by a given response. This system is **non-invertible for complex $x(t)$** .

22. System equation: $y(t) = \text{Re}(x(t))$

Homogeneity:

$x_1(t) = g(t) + jh(t)$, where $g(t)$ and $h(t)$ are both real-valued functions.

$$\Rightarrow y_1(t) = \text{Re}[g(t) + jh(t)] = g(t)$$

$x_2(t) = (K_r + jK_i)[g(t) + jh(t)]$, where K_r, K_i are both real constants.

$$\Rightarrow y_1(t) = \text{Re}[(K_r + jK_i)[g(t) + jh(t)]] = K_r g(t) - K_i h(t)$$

$$\neq (K_r + jK_i)g(t) = (K_r + jK_i)y_1(t)$$

So the system is **not homogeneous**.

Additivity:

$x_1(t) = g_1(t) + jh_1(t)$, where $g_1(t)$ and $h_1(t)$ are both real-valued functions.

$$\Rightarrow y_1(t) = \text{Re}[g_1(t) + jh_1(t)] = g_1(t)$$

$x_2(t) = g_2(t) + jh_2(t)$, where $g_2(t)$ and $h_2(t)$ are both real-valued functions.

$$\Rightarrow y_2(t) = \text{Re}[g_2(t) + jh_2(t)] = g_2(t)$$

$x_3(t) = x_1(t) + x_2(t) = (g_1(t) + g_2(t)) + j(h_1(t) + h_2(t))$

$$\Rightarrow y_3(t) = \text{Re}[x_3(t)] = g_1(t) + g_2(t) = y_1(t) + y_2(t)$$

So the system is **additive**.