

Chapter 4

12. $\sum F = ma$

$$-mg - K_s[y(t) - x(t) - d_o] - K_d[y'(t) - x'(t)] = my''(t) \quad (1)$$

a) When car is at rest, set all derivatives in (1) equal to zero which yields

$$-mg - K_s[y(t) - x(t) - d_o] = 0$$

$$\begin{aligned} y(t) - x(t) &= -\frac{mg}{K_s} + d_o \\ &= -\frac{(1500 \text{ kg})(9.81 \frac{\text{m}}{\text{s}^2})}{7500 \text{ N} \cdot \frac{\text{s}}{\text{m}}} + 0.6 \text{ m} \\ &= 0.404 \text{ m} \end{aligned}$$

b) Rearrange (1) as

$$my''(t) + K_d[y'(t) - x'(t)] + K_s[y(t) - x(t) - d_o + \frac{mg}{K_s}] = 0 \quad (2)$$

Define new variable as $z(t) = y(t) - x(t) - d_o + \frac{mg}{K_s}$

$$z'(t) = y'(t) - x'(t)$$

$$z''(t) = y''(t) - x''(t)$$

Substitute into (2) which yields

$$m[z''(t) + x''(t)] + K_d z'(t) + K_s z(t) = 0$$

$$mz''(t) + K_d z'(t) + K_s z(t) = -mx''(t)$$

$$z''(t) + \frac{K_d}{m} z'(t) + \frac{K_s}{m} z(t) = -x''(t) \quad (3)$$

Case 2: $m=n$, add impulse term to general solution

Solve for the homogenous equation

$$z''(t) + \frac{K_d}{m} z'(t) + \frac{K_s}{m} z(t) = 0$$

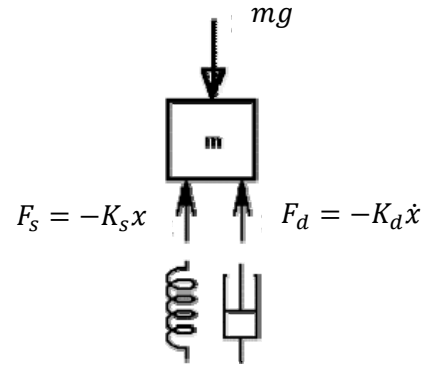
$$\lambda^2 + \frac{K_d}{m} \lambda + \frac{K_s}{m} = 0 \rightarrow \lambda_{1,2} = -\frac{K_d \pm \sqrt{K_d^2 - 4K_s}}{2m} = -6.667 \pm 2.357i$$

$$z(t) = e^{-6.667t} [K_1 \cos(2.357t) + K_2 \sin(2.357t)]$$

$$h_z(t) = e^{-6.667t} [K_1 \cos(2.357t) + K_2 \sin(2.357t)] u(t)$$

The impulse response will take the form of

$$h(t) = e^{-6.667t} [K_1 \cos(2.357t) + K_2 \sin(2.357t)] u(t) + K_\delta \delta(t)$$



Method 1:

Solve simplified problem first

$$z''(t) + \frac{K_d}{m} z'(t) + \frac{K_s}{m} z(t) = x(t)$$

$$z'(0^+) - z'(0^-) + \frac{K_d}{m} [z(0^+) - z(0^-)] + \frac{K_s}{m} \int_{0^-}^{0^+} z(t) dt = \int_{0^-}^{0^+} x(t) dt = 1$$

$$z'(0^+) + \frac{K_d}{m} z(0^+) = 1 \quad \rightarrow \quad -6.667K_1 + 2.357K_2 + \frac{K_d}{m} K_1 = 1$$

$$z(0^+) - z(0^-) + \frac{K_d}{m} \int_{0^-}^{0^+} z(t) dt = 0$$

$$z(0^+) = 0 \quad \rightarrow \quad K_1 = 0$$

Thus $K_1 = 0$ $K_2 = 0.424$ which yields the homogenous solution as

$$h_z(t) = [0.424e^{-6.667t} \sin(2.357t)]u(t)$$

Then define new variable $g(t)$ where $b_0 = -1$ to solve for (3)

$$g(t) = z(t) - b_0 x(t)$$

$$g(t) = z(t) + x(t)$$

$$g'(t) = z'(t) + x'(t)$$

$$g''(t) = z''(t) + x''(t)$$

Substitute into (3) which yields

$$g''(t) - x''(t) + \frac{K_d}{m} [g'(t) - x'(t)] + \frac{K_s}{m} [g(t) - x(t)] = -x''(t)$$

$$g''(t) + \frac{K_d}{m} g'(t) + \frac{K_s}{m} g(t) = \frac{K_s}{m} x(t) + \frac{K_d}{m} x'(t)$$

Set $x(t) = h_z(t)$ to solve for homogenous solution of (3)

$$h_g(t) = \left[\frac{K_s}{m} x(t) + \frac{K_d}{m} x'(t) \right] u(t) = \left[\frac{K_s}{m} h_z(t) + \frac{K_d}{m} h_z'(t) \right] u(t)$$

$$h_g(t) = \left\{ \frac{K_s}{m} [0.424e^{-6.667t} (\sin(2.357t))] + \frac{K_d}{m} [-2.826e^{-6.667t} \sin(2.357t) + e^{-6.667t} \cos(2.357t)] \right\} u(t)$$

$$h_g(t) = \{ 50[0.424e^{-6.667t} (\sin(2.357t))] + 13.333[-2.826e^{-6.667t} \sin(2.357t) + e^{-6.667t} \cos(2.357t)] \} u(t)$$

$$h_g(t) = e^{-6.667t} [13.333 \cos(2.357t) - 16.479 \sin(2.357t)] u(t)$$

$$h(t) = h_g(t) - b_0\delta(t)$$

Therefore the impulse response is

$$h(t) = e^{-6.667t}[13.333 \cos(2.357t) - 16.479 \sin(2.357t)]u(t) - \delta(t)$$

Method 2:

Let $z(t) = h(t)$ and $x(t) = \delta(t)$ for (3), then

$$h''(t) + \frac{K_d}{m}h'(t) + \frac{K_s}{m}h(t) = -\delta''(t)$$

Integrate from $t = 0^+$ to $t = 0^-$:

- (i)
$$h'(0^+) + h'(0^-) + \frac{K_d}{m}[h(0^+) + h(0^-)] + \frac{K_s}{m} \int_{0^-}^{0^+} h(t)dt = -[\delta'(0^+) - \delta'(0^-)]$$

$$\Rightarrow h'(0^+) + \frac{K_d}{m}h(0^+) + \frac{K_s}{m} \int_{0^-}^{0^+} h(t)dt = 0$$

$$\Rightarrow (-6.667K_1 + 2.357K_2) + \frac{K_d}{m}K_1 + \frac{K_s}{m}K_\delta = 0 \quad (a)$$
- (ii)
$$h(0^+) + h(0^-) + \frac{K_d}{m} \int_{0^-}^{0^+} h(t)dt + \frac{K_s}{m} \int_{0^-}^{0^+} \int_{-\infty}^t h(t)dt = -[\delta(0^+) - \delta(0^-)]$$

$$\Rightarrow K_1 + \frac{K_d}{m}K_\delta = 0 \quad (b)$$
- (iii)
$$\int_{0^-}^{0^+} h(t)dt + \frac{K_d}{m} \int_{0^-}^{0^+} \int_{-\infty}^t h(t)dt + \frac{K_s}{m} \int_{0^-}^{0^+} \int_{-\infty}^t \int_{-\infty}^s h(s)dsdt = -\int_{0^-}^{0^+} \delta(t)dt$$

$$\Rightarrow K_\delta = -1 \quad (c)$$

Solving equations (a) (b) (c) we get $K_1 = 13.333, K_2 = 16.497, K_\delta = -1$

Therefore the impulse response is

$$h(t) = e^{-6.667t}[13.333 \cos(2.357t) - 16.479 \sin(2.357t)]u(t) - \delta(t)$$

- c) Apply convolution integral $h_c = 0.15 m$

$$z(t) = 0.15h(t) * u(t) = 0.15 \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$

$$z(t) = 0.15 \int_{-\infty}^{\infty} \{e^{-6.67\tau}[13.33 \cos(2.36\tau) - 16.5 \sin(2.36\tau)]u(\tau) - \delta(\tau)\}u(t-\tau)d\tau$$

Use following properties to solve for $z(t)$

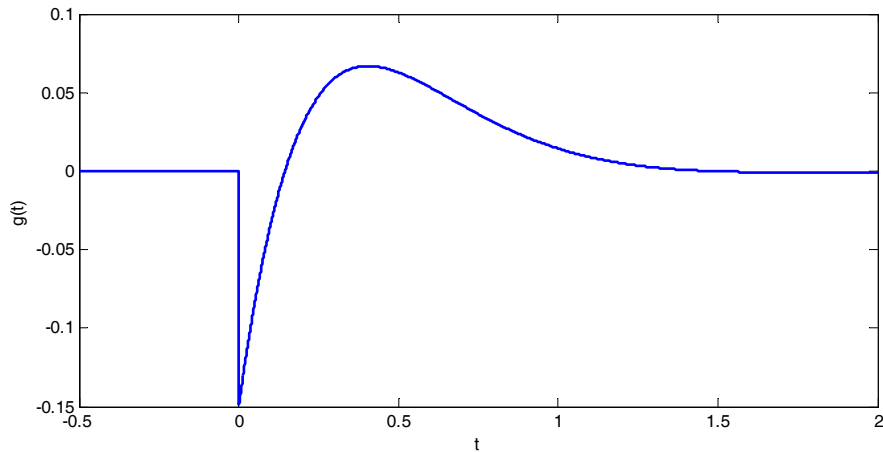
$$\int_0^t e^{ax} \sin(bx) dx = \frac{e^{ax}}{a^2 + b^2} [a \sin(bx) - b \cos(bx)]$$

$$\int_0^t e^{ax} \cos(bx) dx = \frac{e^{ax}}{a^2 + b^2} [a \cos(bx) + b \sin(bx)]$$

$$z(t) = 0.15 \left\{ 13.33 \frac{e^{-6.67t}}{50} [-6.67 \cos(2.36t) + 2.36 \sin(2.36t)] - 16.5 \frac{e^{-6.67t}}{50} [-6.67 \sin(2.36t) - 2.36 \cos(\tau)] \right\} - 0.15u(t)$$

$$z(t) = 0.15\{e^{-3.33t}[2.81 \sin(2.36t) - \cos(2.36t)] + 1\}u(t) - 0.15u(t)$$

$$z(t) = 0.15e^{-3.33t}[2.81 \sin(2.36t) - \cos(2.36t)]u(t)$$



20. $y(t) = \int_{-\infty}^{t/3} x(\lambda) d\lambda$

Time-Invariance:

Let $x_1(t) = g(t)$. Then $y_1(t) = \int_{-\infty}^{t/3} g(\lambda) d\lambda$.

Let $x_2(t) = g(t - t_0)$. Then $y_2(t) = \int_{-\infty}^{t/3} g(\lambda - t_0) d\lambda = \int_{-\infty}^{t/3 - t_0} g(u) du$

Substitute t with $t - t_0$ into $y_1(t)$ yields $y_1(t - t_0) = \int_{-\infty}^{(t - t_0)/3} g(\lambda) d\lambda$

$y_2(t) \neq y_1(t - t_0)$, thus the system is **time-variant**.

Stability:

If $x(t)$ is bounded, then $y(t) = \int_{-\infty}^{t/3} x(\lambda) d\lambda = x(\lambda) \int_{-\infty}^{t/3} d\lambda$ increases without bound therefore the system is **unstable**.

Invertibility:

$$y'(t) = x\left(\frac{t}{3}\right) \quad \rightarrow \quad x(t) = y'(3t)$$

Thus the system is **invertible**.

Chapter 6

18c. $-y''(t) + 3y'(t) = 3x(t) + 5x''(t)$

Rewrite as $y''(t) - 3y'(t) = -3x(t) - 5x''(t)$ (1)

Case 2: $m=n$, add impulse term to general solution

Solve for the homogeneous solution

$$y''(t) - 3y'(t) = 0$$

$$\lambda^2 - 3\lambda = 0 \quad \rightarrow \quad \lambda_{1,2} = 0, 3$$

$$y(t) = K_1 + K_2 e^{3t}$$

$$h_z(t) = (K_1 + K_2 e^{3t})u(t)$$

Method 1:

Solve simplified problem first

$$y''(t) - 3y'(t) = x(t)$$

$$y'(0^+) - y'(0^-) - 3[y(0^+) - y(0^-)] = \int_{0^-}^{0^+} x(t)dt = 1$$

$$y'(0^+) - 3y(0^+) = 1 \quad \rightarrow \quad 3K_2 - 3(K_1 + K_2) = 1$$

$$y(0^+) - y(0^-) - 3 \int_{0^-}^{0^+} y(t)dt = \int_{0^-}^{0^+} \int_{-\infty}^t x(s)dsdt = 0$$

$$y(0^+) = 0 \quad \rightarrow \quad K_1 + K_2 = 0$$

$$\text{Thus, } K_1 = -\frac{1}{3} \quad K_2 = \frac{1}{3}$$

$$h_z(t) = \left[-\frac{1}{3}(1 - e^{3t}) \right] u(t)$$

Define new variable $g(t)$ where $b_0 = -5$ to solve for (1)

$$g(t) = y(t) - b_0 x(t)$$

$$g(t) = y(t) + 5x(t)$$

$$g'(t) = y'(t) + 5x'(t)$$

$$g''(t) = y''(t) + 5x''(t)$$

Substitute into (1) which yields

$$g''(t) - 5x''(t) - 3[g'(t) - 5x'(t)] = -3x(t) - 5x''(t)$$

$$g''(t) - 3g'(t) = -3x(t) - 15x'(t)$$

$$h_g(t) = [-3h_z - 15h'_z(t)]u(t)$$

$$h_g(t) = \left[-3\frac{1}{3}(1 - e^{3t}) - 15e^{3t} \right] u(t) = (1 - 16e^{3t})u(t)$$

$$h(t) = h_g(t) - b_0 \delta(t)$$

Therefore the impulse response is

$$h(t) = (1 - 16e^{3t})u(t) - 5\delta(t)$$

Method 2:

Let $y(t) = h(t)$ and $x(t) = \delta(t)$ for (1), then

$$h''(t) - 3h'(t) = -3\delta(t) - 5\delta''(t)$$

Integrate from $t = 0^+$ to $t = 0^-$:

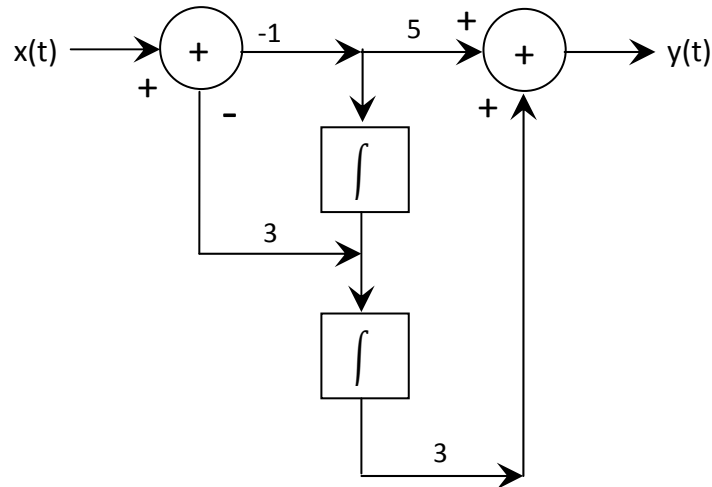
$$\begin{aligned} \text{(i)} \quad & -[h'(0^+) - h'(0^-)] + 3[h(0^+) - h(0^-)] = 3 \int_{0^-}^{0^+} \delta(t) dt + 5[\delta'(0^+) - \delta'(0^-)] \\ & \Rightarrow -h'(0^+) + 3h(0^+) = 3 \int_{0^-}^{0^+} \delta(t) dt \quad \rightarrow \quad -3K_2 + 3(K_1 + K_2) = 3 \quad \text{(a)} \\ \text{(ii)} \quad & -[h(0^+) - h(0^-)] + 3 \int_{0^-}^{0^+} h(t) dt = 3 \int_{0^-}^{0^+} \int_{-\infty}^t \delta(s) ds dt + 5[\delta(0^+) - \delta(0^-)] \\ & \Rightarrow -h(0^+) + 3 \int_{0^-}^{0^+} h(t) dt = 0 \quad \rightarrow \quad -(K_1 + K_2) + 3K_\delta = 0 \quad \text{(b)} \\ \text{(iii)} \quad & -\int_{0^-}^{0^+} h(t) dt + 3 \int_{0^-}^{0^+} \int_{-\infty}^t h(s) ds dt = 3 \int_{0^-}^{0^+} \int_{-\infty}^t \int_{-\infty}^s \delta(w) dw ds dt + 5 \int_{0^-}^{0^+} \delta(t) dt \\ & \Rightarrow -\int_{0^-}^{0^+} h(t) dt = 5 \int_{0^-}^{0^+} \delta(t) dt \quad \rightarrow \quad -K_\delta = 5 \quad \text{(c)} \end{aligned}$$

Solving equations (a) (b) (c) we get $K_1 = 1, K_2 = -16, K_\delta = -5$

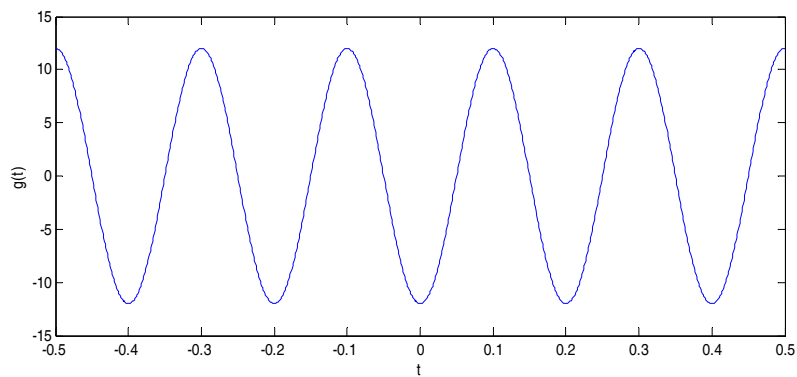
Therefore the impulse response is

$$h(t) = (1 - 16e^{3t})u(t) - 5\delta(t)$$

Plot II-type block diagram with integrators:

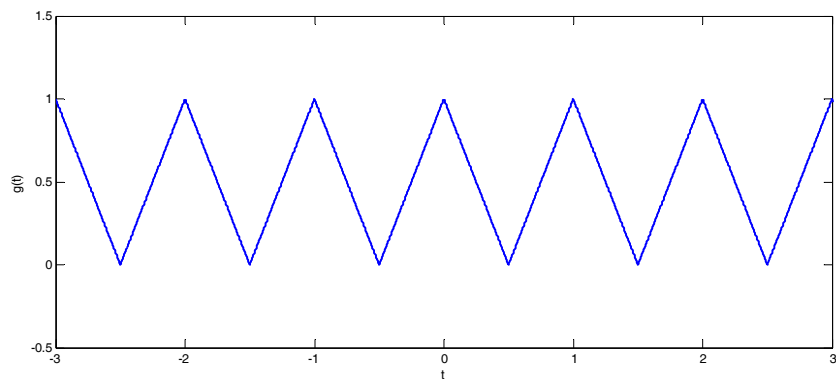


$$\begin{aligned} 22a. \quad g(t) &= 3 \cos(10\pi t) * 4\delta(t + 1/10) \\ &= 12 \cos[10\pi(t + 1/10)] = 12 \cos(10\pi t + \pi) = -12 \cos(10\pi t) \end{aligned}$$



22d. $g(t) = \text{tri}(2t) * \delta_1(t)$

$$= \sum_{n=-\infty}^{\infty} \text{tri}(2(t-n))$$



Chapter 15

33b. $G(s) = \frac{4}{(s+3)(s+8)}$

Express in partial fraction expansion.

$$G(s) = \frac{A}{s+3} + \frac{B}{s+8} = \frac{4}{(s+3)(s+8)}$$

$$A(s+8) + B(s+3) = 4$$

$$s(A+B) + 8A + 3B = 4$$

$$s(A+B) = 0, \quad 8A + 3B = 4 \quad \rightarrow \quad A = \frac{4}{5}, \quad B = -\frac{4}{5}$$

$$G(s) = \frac{\frac{4}{5}}{s+3} - \frac{\frac{4}{5}}{s+8}$$

From the Laplace transform table on pg. 591, we get

$$g(t) = \left(\frac{4}{5}e^{-3t} - \frac{4}{5}e^{-8t} \right) u(t)$$

Initial-value theorem

$$g(0^+) = \lim_{s \rightarrow \infty} sG(s) = \lim_{s \rightarrow \infty} \frac{4s}{(s+3)(s+8)} = 0$$

Final-value theorem

$$\lim_{t \rightarrow \infty} g(t) = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} \frac{4s}{(s+3)(s+8)} = 0$$

$$33b. \quad G(s) = \frac{s}{s^2 + 2s + 2}$$

$$G(s) = \frac{s}{(s+1)^2 + 1} = \frac{s+1}{(s+1)^2 + 1} + \frac{1}{(s+1)^2 + 1}$$

From the Laplace transform table on pg. 591, we get

$$g(t) = e^{-t}[\cos(t) - \sin(t)]u(t)$$

Initial-value theorem

$$g(0^+) = \lim_{s \rightarrow \infty} sG(s) = \lim_{s \rightarrow \infty} \frac{s^2}{s^2 + 2s + 2} = 1$$

Final-value theorem

$$\lim_{t \rightarrow \infty} g(t) = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} \frac{s^2}{s^2 + 2s + 2} = 0$$

$$39a) \quad y''(t) + 2y'(t) + 10y(t) = x(t)$$

First Laplace transform both sides of the equation.

$$s^2Y(s) - sy(0^-) - \frac{d}{dt}(y(t))_{t=0^-} + 2[sY(s) - y(0^-)] + 10Y(s) = \frac{1}{s}$$

Then solve for Y(s).

$$Y(s) = \frac{\frac{1}{s} + sy(0^-) + 2y(0^-) + \frac{d}{dt}(y(t))_{t=0^-}}{s^2 + 2s + 10}$$

$$Y(s) = \frac{\frac{1}{s} - 5s}{s^2 + 2s + 10} = \frac{1}{s} \frac{1 - 5s^2}{(s+1)^2 + 9}$$

Express in partial fraction expansion.

$$Y(s) = \frac{A}{s} + \frac{Bs + C}{(s+1)^2 + 9}$$

$$A(s+1)^2 + 9A + s(Bs + C) = 1 - 5s^2$$

$$As^2 + 2As + A + 9A + Bs^2 + Cs = 1 - 5s^2$$

$$(A+B)s^2 + (2A+C)s + 10A = 1 - 5s^2$$

$$A+B = -5, 2A+C = 0, 10A = 1 \rightarrow A = \frac{1}{10}, B = -\frac{51}{10}, C = -\frac{1}{5}$$

$$Y(s) = \frac{\frac{1}{10}}{s} + \frac{-\frac{51}{10}s - \frac{1}{5}}{(s+1)^2 + 9} = \frac{1}{10} \left(\frac{1}{s} - \frac{51s+2}{(s+1)^2 + 9} \right)$$

$$Y(s) = \frac{1}{10} \left[\frac{1}{s} - 51 \frac{s+1}{(s+1)^2 + 9} + \frac{49}{3} \frac{3}{(s+1)^2 + 9} \right]$$

From the Laplace transform table on pg. 591, we get

$$y(t) = \frac{1}{10} \left[1 - e^{-t} \left(51 \cos(3t) - \frac{49}{3} \sin(3t) \right) \right] u(t)$$

Additional Item:

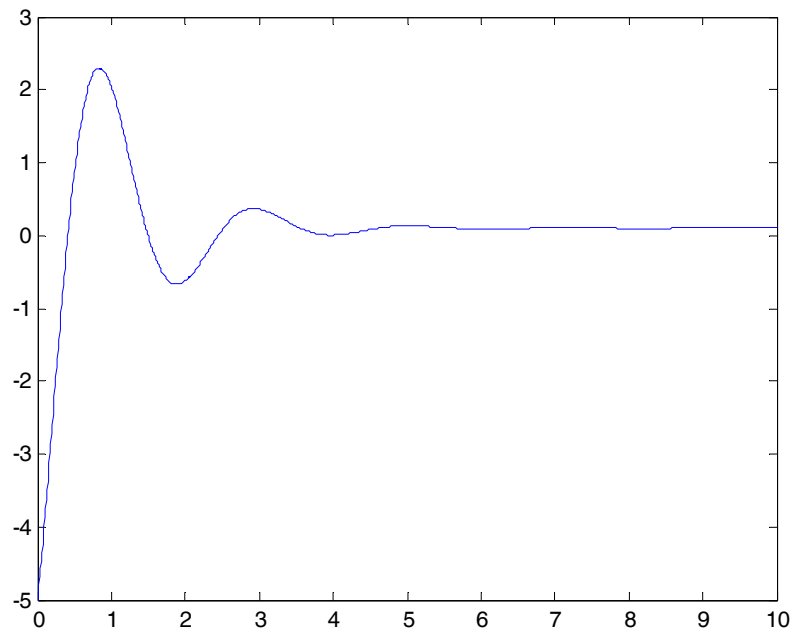
% Problem 39 (a): Use matlab to compute and plot the response of the
% system. You should use commands like 'tf', and 'step'.

% Method 1: use 'tf' and 'step' command - simulate response at initial
% conditions and add to step response

```
Y = tf([1],[1 2 10]); % construct transfer function
a = [0 1; -10 -2]; b = [0; 1]; c = [1 0]; d = [0];
Y1 = ss(a,b,c,d); % construct state space
y0 = [-5 10]; % initial conditions
t = 0:0.01:10;
y1 = step(Y,t); % step response at zero state
y2 = initial(Y1,y0,t); % initial condition response
y = y1+y2; % time response with I.C.s
plot(t,y);
```

% Method 2: use 'lsim' command - compute time response to zero inputs

```
Y = tf([1],[1 2 10]); % construct transfer function
a = [0 1; -10 -2]; b = [0; 1]; c = [1 0]; d = [0];
Y1 = ss(a,b,c,d); % construct state space
x = sign(t); % x(t) = u(t)
y0 = [-5 10]; % initial conditions
t = 0:0.01:10;
y = lsim(Y1,x,t,y0); % time response with I.C.s
plot(t,y);
```



43. $H_1 = \frac{K}{(s+1)(s+2)} \quad H_2 = 1$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1}{1 + H_1 H_2}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{K}{(s+1)(s+2)}}{1 + \frac{K}{(s+1)(s+2)}}$$

$$H(s) = \frac{K}{s^2 + 3s + 2 + K}$$

Find the roots of the denominator

$$s = \frac{-3 \pm \sqrt{9 - 4(2 + K)}}{2} = \frac{-3 \pm \sqrt{1 - 4K}}{2}$$

The system is stable for **all positive real values of K**.

Additional Item:

`% Problem 43: For a fixed value of K that makes the system
% stable, use the matlab commands 'tf' and 'feedback' to compute
% the transfer function from x to y. Use 'impz' to plot the
% impulse response of the resulting system.`

`% Solution: set K to any real positive value, i.e. K = 1`

```
num = [1];  
den = [1 3 2];  
H = tf(num,den)  
Hf = feedback(H,1)  
impulse(Hf)
```

```
>> H = tf(num,den)
```

Transfer function:

1

$s^2 + 3s + 2$

```
>> Hf = feedback(H,1)
```

Transfer function:

1

$s^2 + 3s + 3$

