

Continuous-time Systems

(AKA analog systems)

Recall course objectives

Main Course Objective:

Fundamentals of systems/signals interaction

(we'd like to understand how systems transform or affect signals)

Specific Course Topics:

- Basic test signals and their properties
- Systems and their properties
- Signals and systems interaction
 - Time Domain: convolution
 - Frequency Domain: frequency response
- Signals & systems applications:
 - audio effects, filtering, AM/FM radio
- Signal sampling and signal reconstruction

II. CT systems and their properties

Goals

I. A first classification of systems and their models:

A. Operator Systems: maps that act on signals

B. Physical Systems: examples and ODE models

II. Classification of systems according to their properties:

Homogeneity, time invariance, superposition, linearity,
memory, BIBO stability, controllability, invertibility, ...

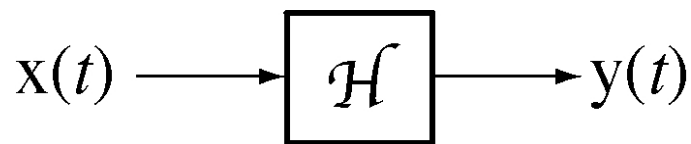
Systems

Systems accept *excitations or input signals* and produce *responses or output signals*

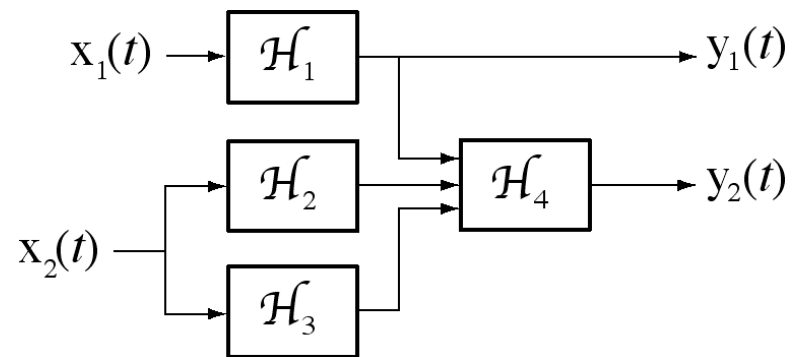
Systems are often represented by block diagrams

SISO = single input, single output

MIMO = multiple input, multiple output



SISO system



MIMO system

Operator systems acting on signals

Systems can be “operators” or “maps” that combine signals

This is a more usual situation in the “digital world.” However some analog systems can also be described through maps

Examples:

1-Algebraic operators

- noise removal by **averaging**
- motion detection through image **subtraction**

2-Geometric/point operators (interpolation)

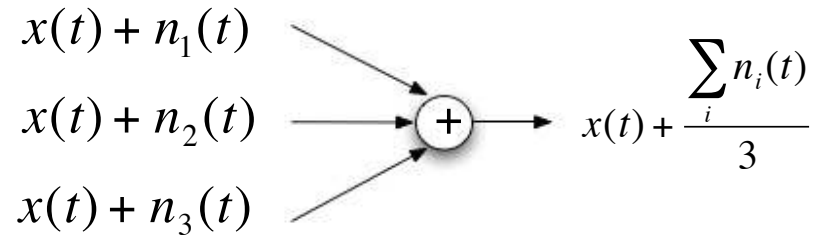
- image **rectification**
- visual spatial effects: morphing, image transformation
- cartography: maps of ellipsoidal/spherical bodies

3-Signal multiplication:

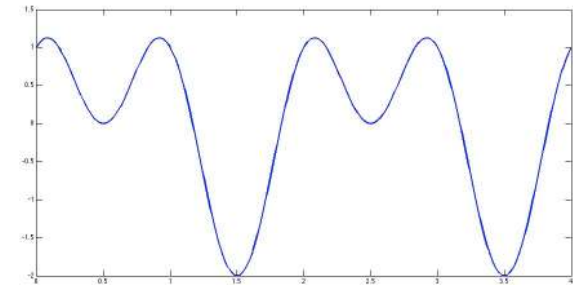
- AM radio signal modulation before transmission

However, simple operations like these are not enough to capture all filtering effects on signals (more on this later)

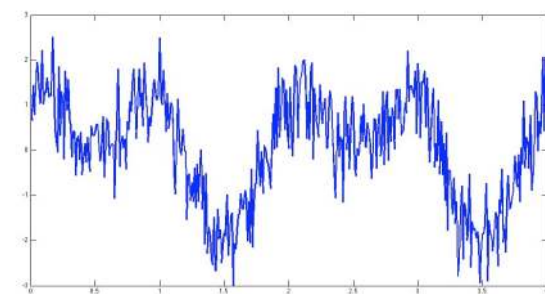
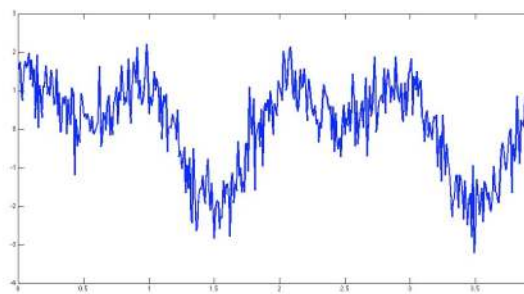
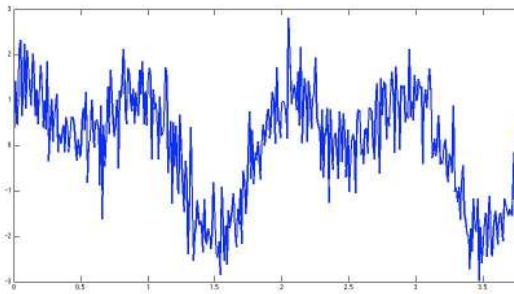
Noise* removal by averaging



Original signal



Noisy versions of the signal (noise is zero mean)

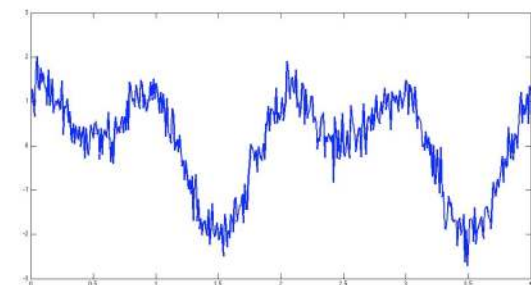


Period of underlying signal
must be known or estimated

Matlab script in webpage: averaging.m

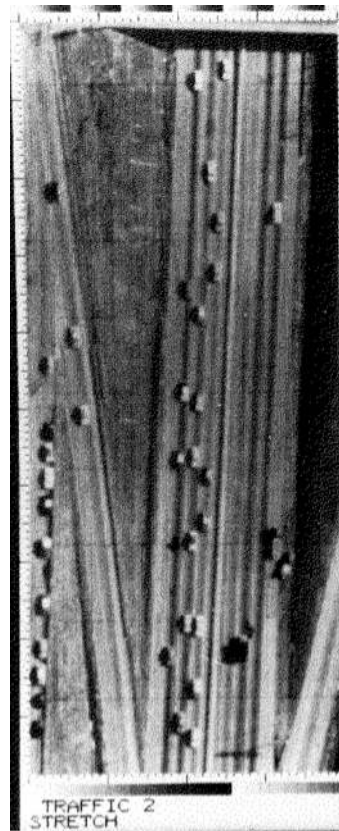
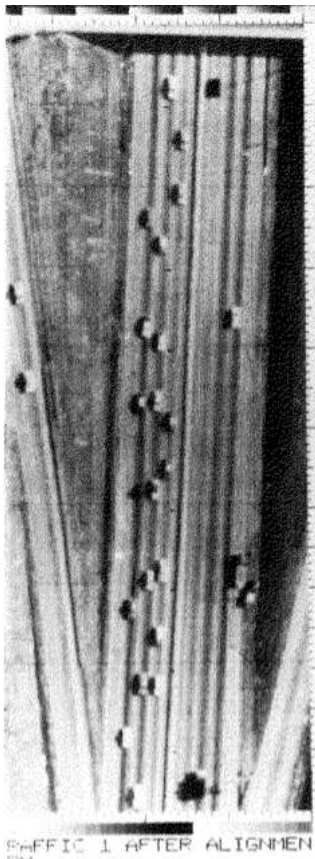
(*) noise = infinite-energy signal
that takes random values

Averaged signal



Motion detection by subtraction

$$\begin{array}{c} A_0(x,y) \\ A_1(x,y) \end{array} \rightarrow \begin{array}{c} \ominus \end{array} \rightarrow A_0(x,y) - A_1(x,y)$$



(this is a digital system example: here image signals are a function of discrete-space variables or pixels)

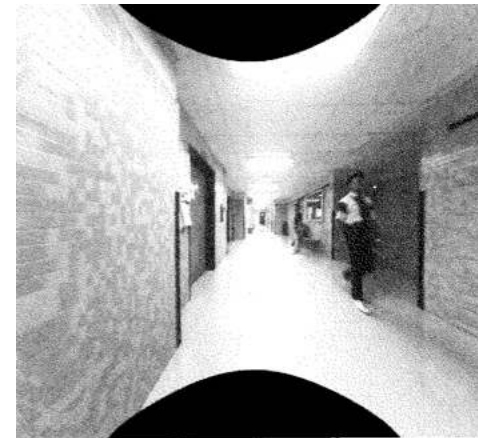
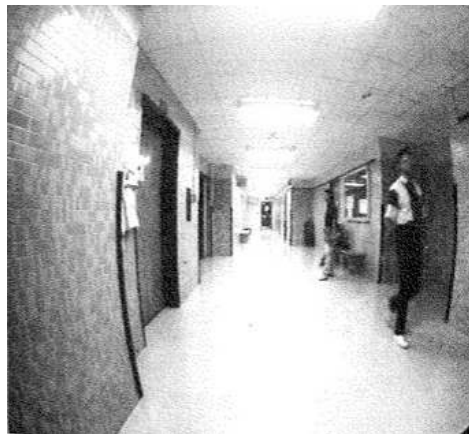
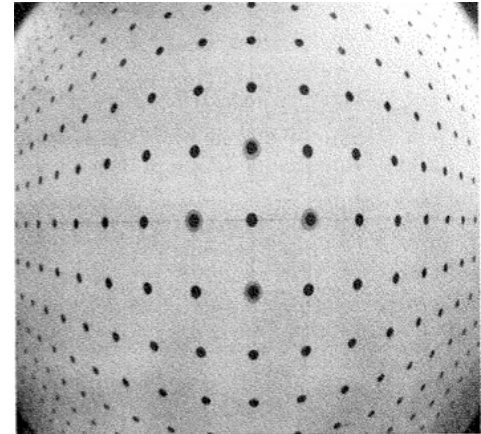
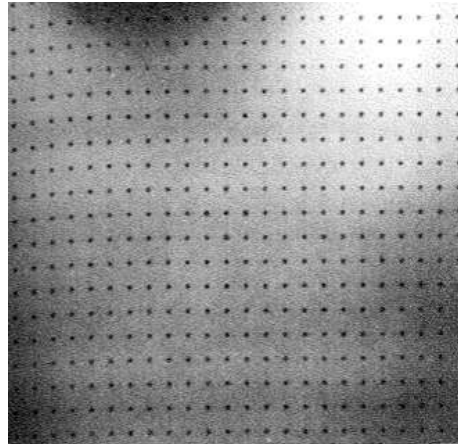
Geometric/point operators

E.g., fish-eye lenses (used in robotics) distort reality

input-output
test using a known
image for camera
calibration
(we'd like to find f :

$$A(x,y) \rightarrow \boxed{f} \rightarrow B(x,y)$$

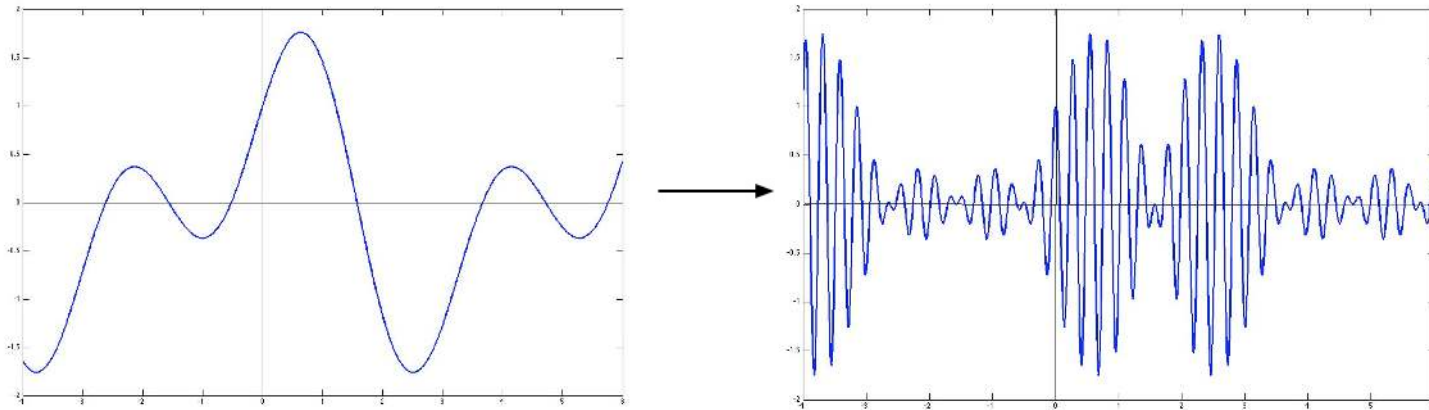
correction
of a real image
using the inferred f



Signal multiplication

AM signal modulation for signal transmission
(used in AM radio)

$$x(t) \rightarrow \begin{array}{c} \text{---} \times \text{---} \\ \uparrow \\ \cos(\omega_c t) \end{array} \rightarrow x(t)\cos(\omega_c t)$$



II. CT systems and their properties

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A. Operator Systems: maps that act on signals

B. Physical Systems: ODE models and examples

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Homogeneity, time invariance, superposition, linearity,
memory, BIBO stability, invertibility, controllability

ODEs and state-space system models

n dimensional ODEs(*) model many electro-mechanical systems

$$\frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) = b_0 \frac{d^m U(t)}{dt^m} + \dots + b_{m-1} \frac{dU(t)}{dt} + b_m U(t)$$

The coefficients a_i, b_i can be time-varying or constant.
When independent of $y(t)$, ODE is linear, otherwise
ODE is nonlinear

We will typically consider $b_i = 0$ for $i = 0, \dots, m-1$

To solve the ODE we need to fix some initial conditions

$$y(t_0), \frac{dy}{dt}(t_0), \frac{d^2 y}{dt^2}(t_0), \dots, \frac{d^{n-1} y}{dt^{n-1}}(t_0)$$

(*) the unknowns are the $y(t)$ and its derivatives

ODEs and state-space system models

Given:

$$\frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) = b_0 \frac{d^m U(t)}{dt^m} + \dots + b_{m-1} \frac{dU(t)}{dt} + b_m U(t)$$

Possible inputs: linear combination of $U(t)$ and derivatives
(represent known variables, e.g., forces in a Mech. System)

Possible outputs: linear combination of $y(t)$ and/or its
derivatives (represent unknown variables we would like to
determine)

$$z_l(t) = c_{l,0} \frac{d^n y(t)}{dt^n} + \dots + c_{l,n-1} \frac{dy(t)}{dt} + c_{l,n} y(t) \quad l = 1, \dots, n$$

When coefficients are independent of $y(t)$, the output equations
are linear; otherwise, they are nonlinear

Physical Systems (mechanical)

Newtonian motion

$$\sum F = ma$$

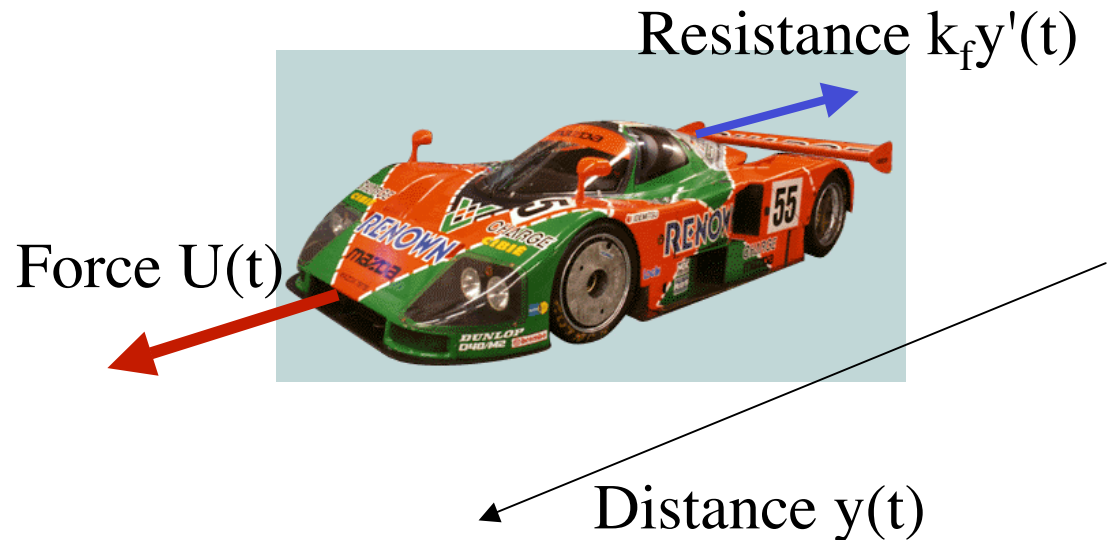
$$\sum \tau = Ia$$

$$M \frac{d^2 y(t)}{dt^2} = -k_f \frac{dy(t)}{dt} + U(t)$$

Input $U(t)$

Output: $y(t)$

Initial conditions $y(t_0)$, $y'(t_0)$



Second-order linear system

$$\frac{d^2 y(t)}{dt^2} + \frac{k_f}{M} \frac{dy(t)}{dt} = \frac{1}{M} U(t), \quad y(t_0), \quad y'(t_0)$$

Physical Systems (mechanical)

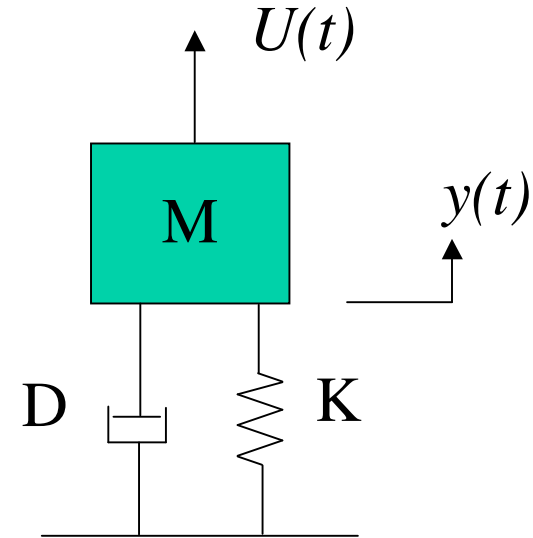
Mass-spring-damper

$$M \frac{d^2 y(t)}{dt^2} + D \frac{dy(t)}{dt} + K y(t) = U(t)$$

Input $U(t)$

Outputs $y(t)$, $y'(t)$, or *combination*

Initial conditions $y(t_0)$, $y'(t_0)$



Second-order linear system

$$\frac{d^2 y(t)}{dt^2} + \frac{D}{M} \frac{dy(t)}{dt} + \frac{K}{M} y(t) = \frac{1}{M} U(t), \quad y(t_0), \quad y'(t_0)$$

Physical Systems (mechanical)

Simple pendulum

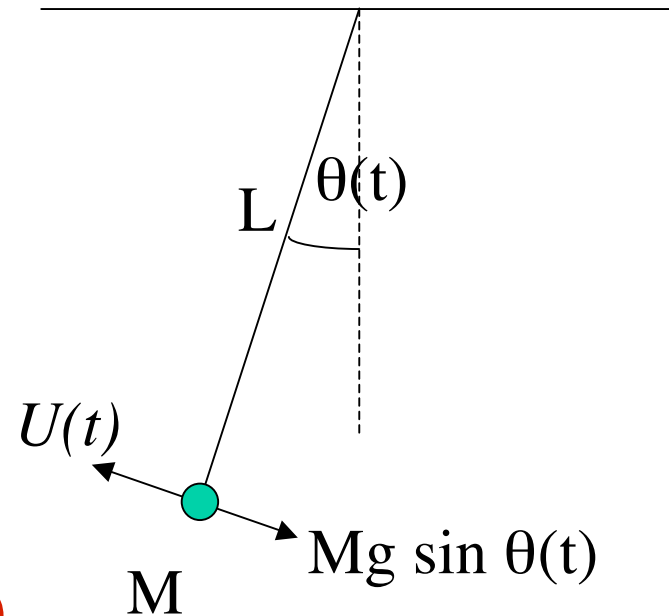
$$I \frac{d^2 \theta(t)}{dt^2} = LU(t) - MgL \sin(\theta)$$

$I = ML^2$, moment of inertia

Input (force at the ball) $U(t)$

Output $\theta(t)$

Initial conditions $\theta(t_0), \theta'(t_0)$



Second-order nonlinear system (why?)

$$\frac{d^2 \theta(t)}{dt^2} + \frac{g}{L} \sin(\theta) = \frac{1}{ML} U(t) \quad \theta(t_0), \theta'(t_0)$$

Linearization for small θ is $\theta''(t) + gL^{-1}\theta(t) = (ML)^{-1}U(t)$

Physical Systems (electrical)

Kirchhoff's laws

Current law:

sum of (signed) currents at a "node" is zero
(node=electrical juncture of two or more devices)

Voltage law:

sum of (signed) voltages around a "loop" is zero
(loop=closed path passing through ordered sequence of nodes)

Circuit element laws:

Resistor:

$$iR = V_R$$



Capacitor:

$$i = C \frac{dV}{dt}$$



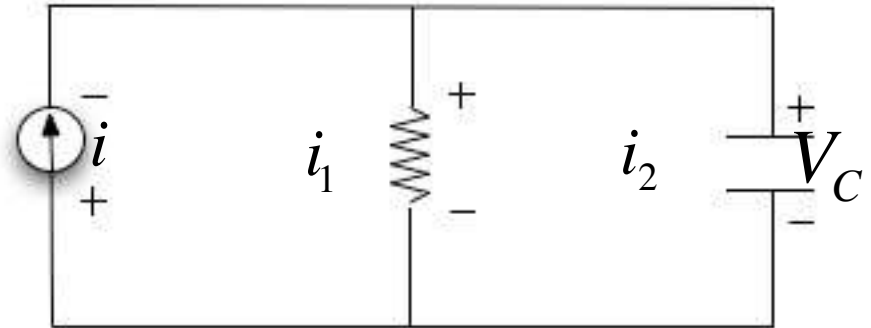
Inductor:

$$V = L \frac{di}{dt}$$



Physical Systems (electrical)

Objective: Find a model relating the input i and output V_C



2 nodes and 2 loops

Equation of upper node: $i - i_1 - i_2 = 0$

Equation of left loop: $V + V_1 = 0$

Equation of right loop: $-V_1 + V_C = 0$

Circuit element equations: $V_1 = i_1 R$ $C \frac{dV_C}{dt} = i_2$

Putting it all together: $i = i_2 + i_1$ $i = C \frac{dV_C}{dt} + \frac{1}{R} V_1$

$$i = C \frac{dV_C}{dt} + \frac{1}{R} V_C$$

ODEs and state-space models

A state-space representation of an nth order ODE describing a physical system is obtained as follows:

State $x = (x_1, x_2, x_3, \dots, x_n)^T = \left(y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \dots, \frac{d^{n-1}y}{dt^{n-1}} \right)^T$

The **state-space representation of the system** is a system of first-order differential equations in the new variables

$$x_1, x_2, x_3, \dots, x_n$$

If the original nth order ODE is linear, then the state-space representation can be expressed in matrix form:

$$\frac{dx}{dt} = Ax + Bu, \quad x(t_0)$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_{n1} & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ b_m \end{pmatrix}$$

A linear output equation is expressed as $z = Cx, \quad C = (c_{l,k})$

This can be generalized for several variables

Why state-space models are used

State-space formulation allows
to lump multiple variables in
a single state vector x

Distillation column:
Hundreds of state variables

Concentration and temp at each
tray position

Lots of structure

Output of one tray is the
input to the next

Several inputs

Boiler power, reflux ratio,
feed rate

Many outputs

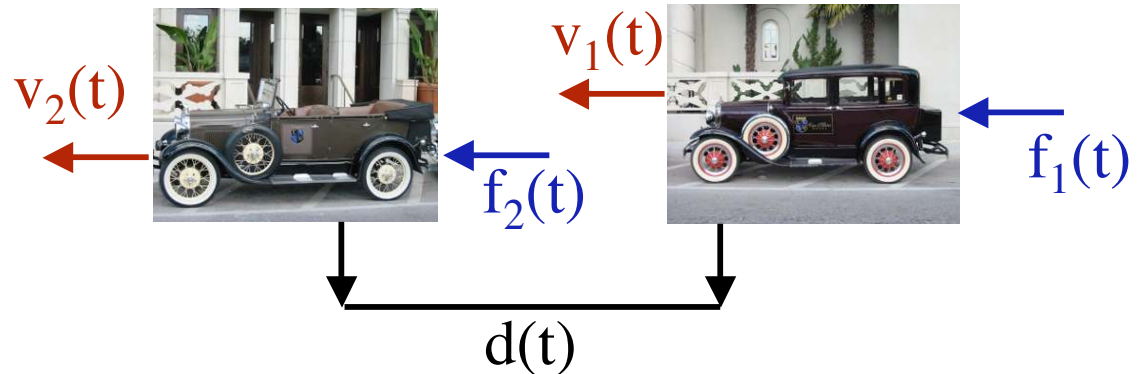
Some tray temperatures,
final concentration



Why state-space models are used

Well suited for MIMO systems

MIMO and SISO systems have same form in state-space formulation



This allows for uniform treatment

- Analysis of system properties
- Linearization
- Simulation (matlab)

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} \frac{-k_{f1}}{M_1} & 0 & 0 \\ 0 & \frac{-k_{f2}}{M_2} & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} M_1^{-1} & 0 \\ 0 & M_2^{-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$$

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ d(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

Why do we linearize about equilibrium points

Unfortunately, there are **no general formulas to solve nonlinear ODEs**. Then we are forced to look for (1) particular solutions and (2) approximations to the solutions

How can we find particular solutions to nonlinear ODEs?

Equilibrium points are always particular constant solutions

How to approximate the solutions of a nonlinear ODE?

(a) We know how to solve linear ODEs

(b) The qualitative behavior of a nonlinear ODE with an initial condition close to an equilibrium point, under inputs of small magnitude, can be found by solving the linearized equation about that equilibrium point with zero inputs

Linearization about equilibrium point

$$\frac{dx}{dt} = f(x,u), \quad x(t_0),$$

An **equilibrium point** is x_0 such that $f(x_0,0) = 0$

Equilibrium point = constant solution to ODE

The system remains at rest at all times if initially placed at the equilibrium and no inputs are applied

Pendulum example. The state is $x = (\theta, \theta')^T$ The pendulum has two equilibrium points:

$x_1 = (0,0)^T$ (vertical bottom position, zero velocity)

$x_2 = (\pi,0)^T$ (vertical top position, zero velocity)

Linearization is easy in state-space formulation

Suppose the map $f : R^n \times R^m \longrightarrow R^n$ is **nonlinear** (as in the pendulum example)

Linearization of the system **about** x_0 **with** $u=0$ is:

$$\frac{dx}{dt} = \left(\frac{\partial f}{\partial x} \right)_{|x=x_0, u=0} (x - x_0) + \left(\frac{\partial f}{\partial u} \right)_{|x=x_0, u=0} u, \quad x(t_0)$$

with constant matrices

$$\left(\frac{\partial f}{\partial x} \right)_{|x=x_0, u=0} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}_{|x=x_0, u=0} \quad \left(\frac{\partial f}{\partial u} \right)_{|x=x_0, u=0} = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{pmatrix}_{|x=x_0, u=0}$$

Linearization with additional output map

For systems with an **additional nonlinear output** map:

$$\frac{dx}{dt} = f(x, u), \quad x(t_0),$$

$$z = h(x),$$

where $h : R^n \longrightarrow R^p$, $h(x_0) = 0$, **linearization** becomes:

$$\frac{dx}{dt} = \left(\frac{\partial f}{\partial x} \right)_{|x=x_0, u=0} (x - x_0) + \left(\frac{\partial f}{\partial u} \right)_{|x=x_0, u=0} u,$$

$$z = \left(\frac{\partial h}{\partial x} \right)_{|x=x_0} (x - x_0)$$

CT systems and their properties

Goals

I System examples and their models e.g. using basic principles

A. Operator systems: maps that act on signals

B. Physical systems: ODE models and examples

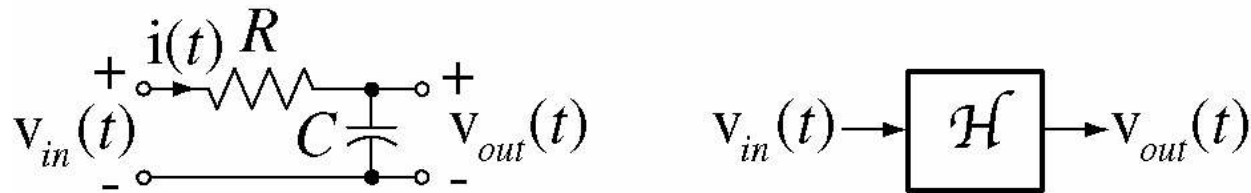
II System properties

Homogeneity, time invariance, superposition, linearity,
memory, invertibility, BIBO stability, controllability

Response of a RC Low-pass filter

An RC low-pass filter is a simple circuit

It can be modeled as a single input, single output system



The system is excited by a voltage $v_{in}(t)$ and responds with a voltage $v_{out}(t)$

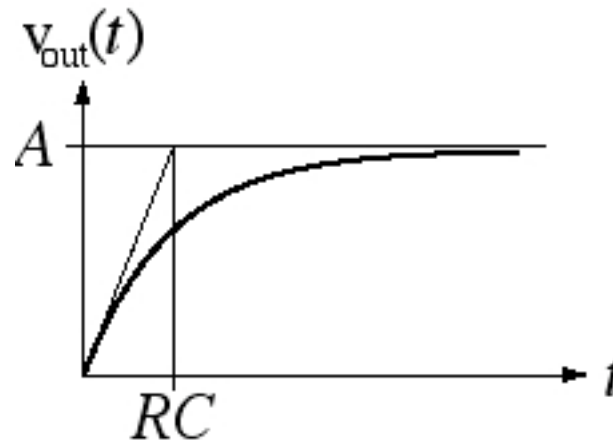
We assume the circuit has no initial energy at the capacitor

Response of a RC Low-pass filter

If the RC low-pass filter is excited by a step voltage

$$v_{in}(t) = A u(t)$$

Its response is

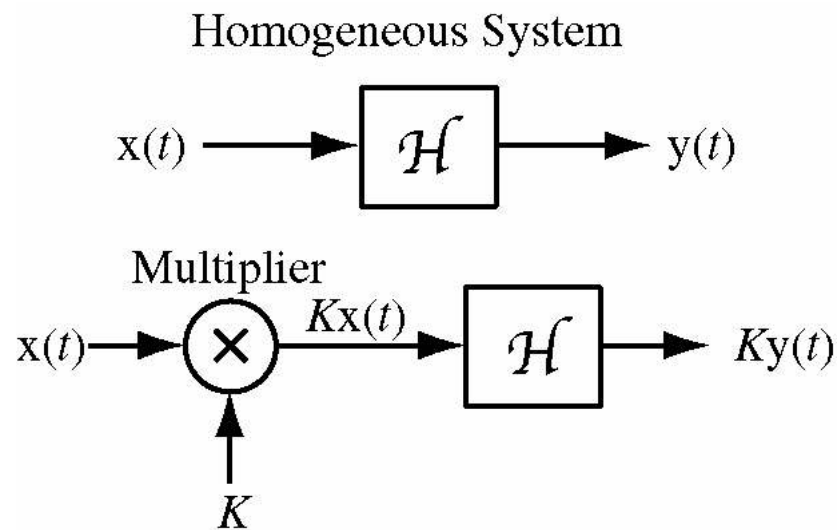


$$v_{out}(t) = A(1 - e^{-t/RC})u(t)$$

That is, if the excitation is doubled, the response doubles

Homogeneity

In a **homogeneous system**, multiplying the excitation by any constant (including *complex constants*), multiplies the response by the same constant.



Homogeneity Test

Homogeneity Test:

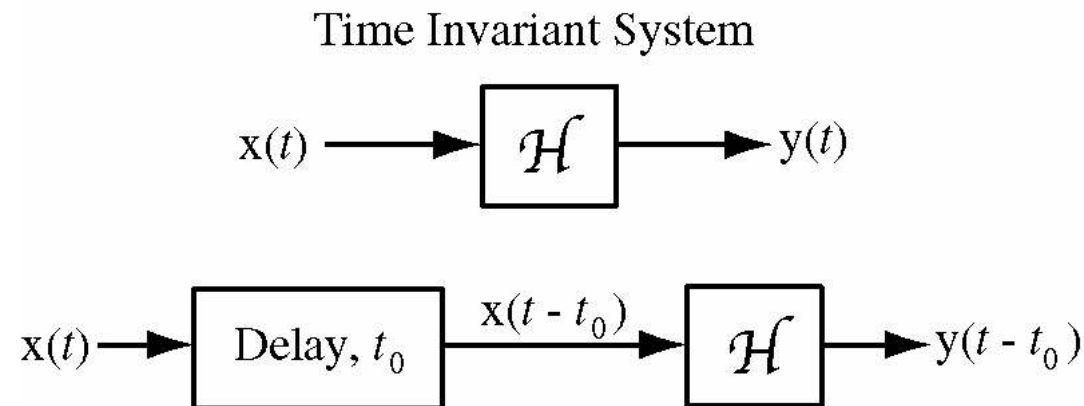
- 1) apply an arbitrary $g(t)$ as input
and obtain $y_1(t)$ as output
- 2) then apply $Kg(t)$ and obtain its output, $h(t)$

If $h(t) = K y_1(t)$ then the system is **homogeneous**

If $g(t) \xrightarrow{\mathcal{H}} y_1(t)$ and $K g(t) \xrightarrow{\mathcal{H}} K y_1(t) \Rightarrow \mathcal{H}$ is Homogeneous

Time invariance

If an excitation causes a response and delaying the excitation simply delays the response by the same amount of time, then **the system is time invariant**

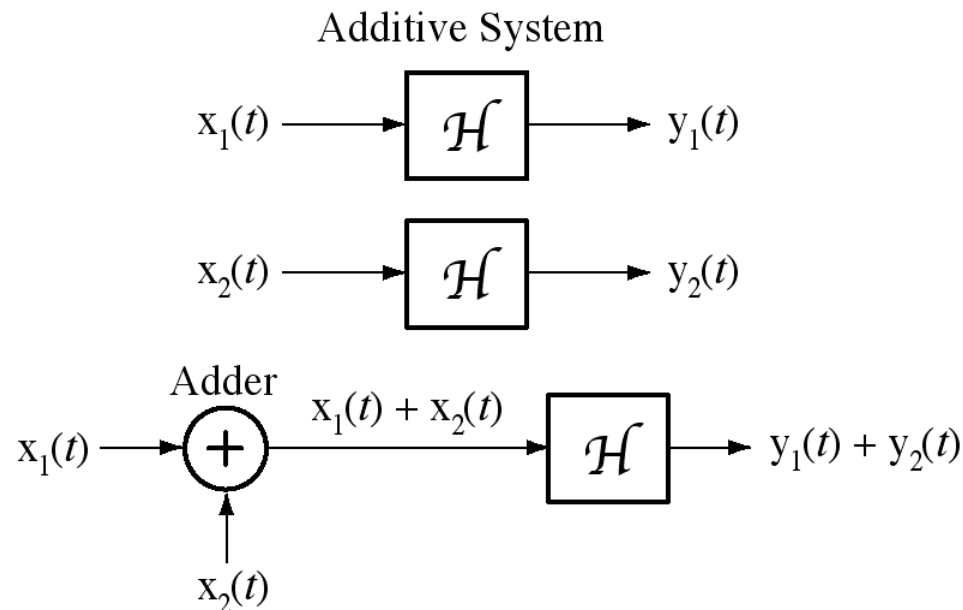


If $g(t) \xrightarrow{\mathcal{H}} y_1(t)$ and $g(t - t_0) \xrightarrow{\mathcal{H}} y_1(t - t_0) \Rightarrow \mathcal{H}$ is Time Invariant

This test must succeed for any g and any t_0 .

Additivity property

If one excitation causes a response and another excitation causes another response and if, for any arbitrary excitations, the sum of the two excitations causes a response which is the sum of the two responses, **the system is said to be additive**



If $g(t) \xrightarrow{\mathcal{H}} y_1(t)$ and $h(t) \xrightarrow{\mathcal{H}} y_2(t)$
and $g(t) + h(t) \xrightarrow{\mathcal{H}} y_1(t) + y_2(t) \Rightarrow \mathcal{H}$ is Additive

Linearity and LTI systems

If a system is both homogeneous and additive, it is **linear**

If a system is both linear and time-invariant, it is called an **LTI (linear, time-invariant)** system

Some systems which are non-linear can be accurately approximated for analytical purposes by a linear system for small excitations (recall the discussion on linearization)

We will mainly focus on LTI systems because we can characterize their response to any signal

System Invertibility

A system is **invertible** if unique excitations produce unique responses

In other words, in an invertible system, knowledge of the response is sufficient to determine the excitation

Any system with input $x(t)$ and output $y(t)$ just described by a linear ODE of the form

$$\frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) = x(t)$$

is invertible. A system with input $x(t)$ and output $z(t)$ described by the operator map $z(t) = \sin(y(t))$ is non-invertible because $\sin(y)$ does not have an inverse.

Memory

This concept reflects the extent to which the present behavior of a system (its outputs) is affected by its past (initial conditions or past values of the inputs)

Physical systems modeled through ODEs have memory: this is associated with the system inability to dissipate energy or redistribute it instantaneously

Example: Think about how a pendulum initially off the vertical winds down to the equilibrium position. The time it takes to do it captures the pendulum memory

If a system is well understood then one can relate its memory to specific properties of the system (e.g. "system stability")

In fact, all filtering methods in signal processing are based on exploiting the memory properties of systems

Memory

A system is said to be **memoryless** if for any time t_1 the output at t_1 depends only on the input at time t_1

Example:

If $y(t) = K u(t)$, then the system is memoryless

If $y(t) = K u(t-1)$, then it has memory

(Operator systems described through static maps are usually memoryless)

Any system that contains a derivative in it has memory; e.g., any system described through an ODE



Stability

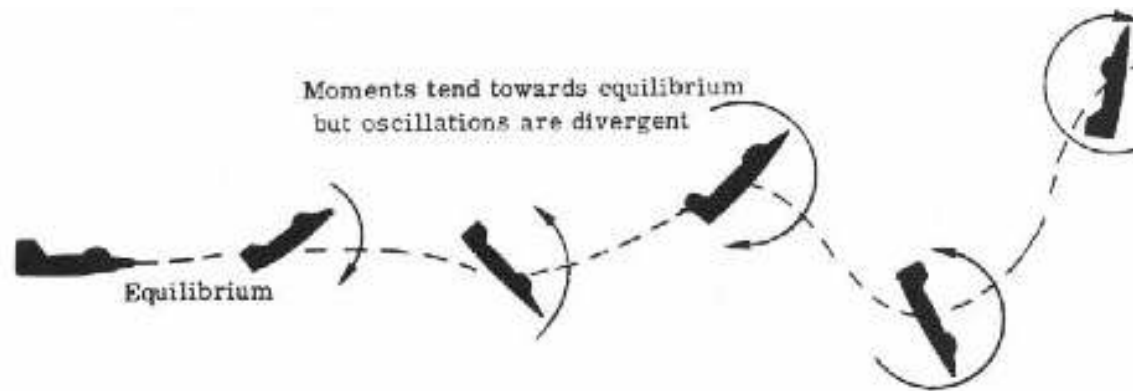
Any system for which the response is bounded for any arbitrary bounded excitation is said to be **bounded-input-bounded-output (BIBO)** stable system, otherwise it is **unstable**

Intuition: All systems for which outputs “do not explode” (I.e. outputs can only reach finite values) are BIBO stable. Intuitively, if a system has a “small memory” or “dissipates energy quickly,” then it will be stable.

If an ODE describing the system is available, then we can apply the following **test for BIBO stability**

A system described by a differential equation is stable if the **eigenvalues** of the solution of the equation all have negative real parts

Stability



Stable systems return to equilibrium despite input disturbances

How to check Stability when ODEs available

Suppose a state-space model for the physical system is available

$$\frac{dx}{dt} = Ax + Bg, \quad x(t_0)$$

Then, the system is stable if and only if the eigenvalues of the matrix A (= the eigenvalues of the ODE) have all negative real parts

The eigenvalues of A are the solutions λ to the equation

$$\det(\lambda I_n - A) = 0$$

Here, n is the dimension of the state x

How to check Stability when ODEs available

Simple example: Low-pass filter

ODE:
$$\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} g(t)$$

state-space representation:

$$\frac{dy(t)}{dt} = -\frac{1}{RC} y(t) + \frac{1}{RC} g(t)$$

Matrix A is just a number:

$$A = -\frac{1}{RC}$$

Calculation of eigenvalues:

$$\det(\lambda \times 1 + \frac{1}{RC}) = 0 \quad \Rightarrow \quad \boxed{\lambda = -\frac{1}{RC}}$$

Thus, the system is BIBO stable

System controllability

A physical system described by a linear state-space model

$$\frac{dx}{dt} = Ax + Bg, \quad x(t_0)$$

is **controllable** if and only if for any initial condition $x(t_0)$ there exists a control $g(t)$ so that we can reach any final state $x(t_f)$ after finite time. That is, if the system is controllable we can do anything with it!

Controllability Theorem: A linear system is controllable if and only if

$$\text{rank}[B, AB, A^2B, A^3B, \dots, A^{n-1}B] = n$$

n is the dimension of the state of the system

Control how? The *Controllability Grammian* is used to find the right $g(t)$ (out of the scope of this course)

Summary

Important points to remember:

1. We **can model** simple (mechanical/electric) systems by resorting to **basic principles** and producing **ODEs**.
2. A special system representation is the **state-space representation**, useful for simulation, linearization and to check system controllability.
3. **Special system properties** are homogeneity, additivity, time-invariance, LTI, invertibility, memory, and stability. These properties can be checked by looking at input-output experiments (no models required in principle.)
4. If an ODE model of the system is available, **we can check system stability** by finding the eigenvalues of the ODE.
5. If an state space representation of the system is available, **we can check the system controllability properties** by applying the controllability theorem.