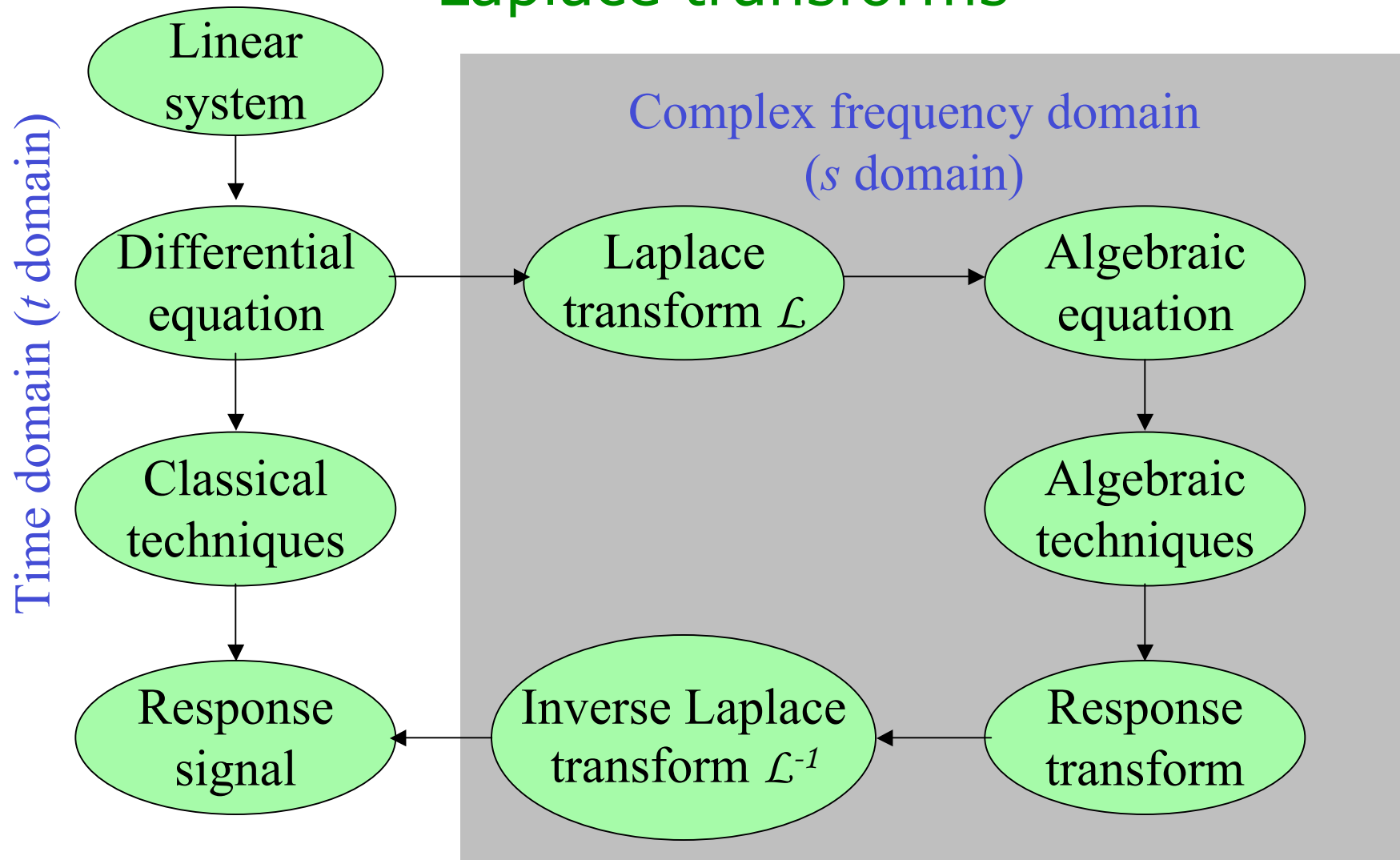


LTI System Analysis with the Laplace Transform

Laplace transforms



The diagram commutes

Same answer whichever way you go

Laplace Transform - definition

Function $f(t)$ of time

Piecewise continuous and exponential order $|f(t)| < Ke^{bt}$

$$F(s) = \int_{0-}^{\infty} f(t)e^{-st} dt$$

$0-$ limit is used to capture transients and discontinuities at $t=0$

s is a complex variable ($\sigma + j\omega$)

There is a need to worry about regions of convergence of the integral

Units of s are $\text{sec}^{-1} = \text{Hz}$

A frequency

Laplace transform examples

Step function – unit Heavyside Function

After Oliver Heavyside (1850-1925) $u(t) = \begin{cases} 0, & \text{for } t < 0 \\ 1, & \text{for } t \geq 0 \end{cases}$

$$F(s) = \int_{0-}^{\infty} u(t)e^{-st} dt = \int_{0-}^{\infty} e^{-st} dt = -\frac{e^{-st}}{s} \Big|_0^{\infty} = -\frac{e^{-(\sigma+j\omega)t}}{\sigma+j\omega} \Big|_0^{\infty} = \frac{1}{s} \quad \text{if } \sigma > 0$$

Exponential function

After Oliver Exponential (1176 BC- 1066 BC)

$$F(s) = \int_0^{\infty} e^{-\alpha t} e^{-st} dt = \int_0^{\infty} e^{-(s+\alpha)t} dt = -\frac{e^{-(s+\alpha)t}}{s+\alpha} \Big|_0^{\infty} = \frac{1}{s+\alpha} \quad \text{if } \sigma > \alpha$$

Delta (impulse) function $\delta(t)$

$$F(s) = \int_{0-}^{\infty} \delta(t)e^{-st} dt = 1 \quad \text{for all } s$$

Laplace Transform Pair Tables

Signal	Waveform	Transform
impulse	$\delta(t)$	1
step	$u(t)$	$\frac{1}{s}$
ramp	$tu(t)$	$\frac{1}{s^2}$
exponential	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
damped ramp	$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$
sine	$\sin(\beta t)u(t)$	$\frac{\beta}{s^2+\beta^2}$
cosine	$\cos(\beta t)u(t)$	$\frac{s}{s^2+\beta^2}$
damped sine	$e^{-\alpha t}\sin(\beta t)u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
damped cosine	$e^{-\alpha t}\cos(\beta t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

Laplace Transform Properties

Linearity – absolutely critical property

Follows from the integral definition

$$\mathcal{L}\{Af_1(t) + Bf_2(t)\} = A\mathcal{L}\{f_1(t)\} + B\mathcal{L}\{f_2(t)\} = AF_1(s) + BF_2(s)$$

Example

$$\begin{aligned}\mathcal{L}\{A\cos(\beta t)\} &= \mathcal{L}\left\{\frac{A}{2}\left[e^{j\beta t} + e^{-j\beta t}\right]\right\} = \frac{A}{2}\mathcal{L}\{e^{j\beta t}\} + \frac{A}{2}\mathcal{L}\{e^{-j\beta t}\} \\ &= \frac{A}{2}\frac{1}{s - j\beta} + \frac{A}{2}\frac{1}{s + j\beta} \\ &= \frac{As}{s^2 + \beta^2}\end{aligned}$$

Laplace Transform Properties

Integration property

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

Proof $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \int_0^\infty \left[\int_0^t f(\tau)d\tau\right] e^{-st} dt$

Denote $x = \frac{-e^{-st}}{s}$, and $y = \int_0^t f(\tau)d\tau$

so $\frac{dx}{dt} = e^{-st}$, and $\frac{dy}{dt} = f(t)$

Integrate by parts $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \left[\frac{-e^{-st}}{s} \int_0^t f(\tau)d\tau\right]_0^\infty + \frac{1}{s} \int_0^\infty f(t)e^{-st} dt$

Laplace Transform Properties

Differentiation Property

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0-)$$

Proof via integration by parts again

$$\begin{aligned}\mathcal{L}\left\{\frac{df(t)}{dt}\right\} &= \int_{0-}^{\infty} \frac{df(t)}{dt} e^{-st} dt = \left[\frac{df(t)}{dt} e^{-st} \right]_{0-}^{\infty} + s \int_{0-}^{\infty} f(t) e^{-st} dt \\ &= sF(s) - f(0-)\end{aligned}$$

Second derivative

$$\begin{aligned}\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} &= \mathcal{L}\left\{\frac{d}{dt} \left[\frac{df(t)}{dt} \right]\right\} = s \mathcal{L}\left\{\frac{df(t)}{dt}\right\} - \frac{df}{dt}(0-) \\ &= s^2 F(s) - sf(0-) - f'(0-)\end{aligned}$$

Laplace Transform Properties

General derivative formula

$$\mathcal{L}\left\{\frac{d^m f(t)}{dt^m}\right\} = s^m F(s) - s^{m-1} f(0-) - s^{m-2} f'(0-) - \dots - f^{(m)}(0-)$$

Translation properties

s -domain translation

$$\mathcal{L}\{e^{-\alpha t} f(t)\} = F(s + \alpha)$$

t -domain translation

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as} F(s) \text{ for } a > 0$$

Laplace Transform Properties

Initial Value Property

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

Final Value Property

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

Caveats:

Laplace transform pairs do not always handle discontinuities properly

Often get the average value

Initial value property no good with impulses

Final value property no good with cos, sin etc

Laplace Transform Properties

Multiplication-Convolution Property

$$\mathcal{L}\{f(t) * g(t)\} = F(s)G(s)$$

Critical property for notion of transfer function

More in a bit...

Rational Functions

We shall mostly be dealing with LTs which are rational functions – ratios of polynomials in s

$$F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$
$$= K \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}$$

p_i are the poles and z_i are the zeros of the function

K is the scale factor or (sometimes) gain

A proper rational function has $n \geq m$

A strictly proper rational function has $n > m$

An improper rational function has $n < m$

How Laplace transforms are born... in MAE 143A

From linear ODEs:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots + a_n y(t) = b_0 x^{(m)}(t) + b_1 x^{(m-1)}(t) + \dots + b_m x(t)$$

Under zero initial conditions, Laplace transforms produce

$$(s^n + a_1 s^{n-1} + \dots + a_n)Y(s) = (b_0 s^m + b_1 s^{m-1} + \dots + b_m)X(s)$$

For any given $X(s)$ we can compute $Y(s)$!

$$Y(s) = H(s)X(s), \quad H(s) := \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{s^n + a_1 s^{n-1} + \dots + a_n}$$

This is the **same as convolution** in the time domain:

$$H(s) = \mathcal{L}\{h(t)\}, \quad Y(s) = H(s)X(s), \quad y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Example

Compute the impulse response of the RC circuit described by the ODE:

$$y'(t) + \frac{1}{RC} y(t) = \frac{1}{RC} x(t)$$

Apply Laplace transform:

$$Y(s) = H(s)X(s), \quad H(s) = \frac{\omega_c}{s + \omega_c}, \quad \omega_c := \frac{1}{RC}$$

The impulse response is then

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{RC} e^{-\frac{1}{RC}t} u(t)$$

Inverting Laplace Transforms

We have a table of inverse LTs

Write $F(s)$ as a partial fraction expansion

$$\begin{aligned} F(s) &= \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \\ &= K \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)} \\ &= \frac{\alpha_1}{(s - p_1)} + \frac{\alpha_2}{(s - p_2)} + \frac{\alpha_{31}}{(s - p_3)} + \frac{\alpha_{32}}{(s - p_3)^2} + \frac{\alpha_{33}}{(s - p_3)^3} + \cdots + \frac{\alpha_q}{(s - p_q)} \end{aligned}$$

Now appeal to linearity to invert via the table

Surprise!

Nastiness: computing the partial fraction expansion is best done by calculating the **residues**

Inverting Laplace Transforms

Compute residues at the poles

$$\lim_{s \rightarrow a} (s - a)F(s) \qquad \frac{1}{(j-1)!} \lim_{s \rightarrow a} \frac{d^{j-1}}{ds^{j-1}} [(s - a)^j F(s)]$$

Bundle complex conjugate pole pairs into second-order terms if you want

$$(s - \alpha - j\beta)(s - \alpha + j\beta) = [s^2 - 2\alpha s + (\alpha^2 + \beta^2)]$$

but you will need to be careful

Inverse Laplace Transform is a sum of complex exponentials

In **Matlab**, check out `[r,p,k]=residue(b,a)`, where

b = coefficients of numerator; a = coefficients of denominator

r = residues; p = poles; k = result of long division

Strictly Proper Laplace Transforms

Find the inverse LT of $F(s) = \frac{20(s+3)}{(s+1)(s^2+2s+5)}$

$$F(s) = \frac{k_1}{s+1} + \frac{k_2}{s+1-2j} + \frac{k_2^*}{s+1+2j}$$

$$k_1 = \lim_{s \rightarrow -1} (s+1)F(s) = \left. \frac{20(s+3)}{s^2+2s+5} \right|_{s=-1} = 10$$

$$k_2 = \lim_{s \rightarrow -1+2j} (s+1-2j)F(s) = \left. \frac{20(s+3)}{(s+1)(s+1+2j)} \right|_{s=-1+2j} = -5-5j = 5\sqrt{2}e^{j\frac{5}{4}\pi}$$

$$\begin{aligned} f(t) &= \left[10e^{-t} + 5\sqrt{2}e^{(-1+2j)t+j\frac{5}{4}\pi} + 5\sqrt{2}e^{(-1-2j)t-j\frac{5}{4}\pi} \right] u(t) \\ &= \left[10e^{-t} + 10\sqrt{2}e^{-t} \cos(2t + \frac{5\pi}{4}) \right] u(t) \end{aligned}$$

Laplace Transforms with Multiple Poles

Compute residues at the poles

$$\frac{1}{(j-1)!} \lim_{s \rightarrow a} \frac{d^{j-1}}{ds^{j-1}} \left[(s-a)^m F(s) \right]$$

Example $\frac{2s^2 + 5s}{(s+1)^3} = \frac{2(s+1)^2 + (s+1) - 3}{(s+1)^3} = \frac{2}{s+1} + \frac{1}{(s+1)^2} - \frac{3}{(s+1)^3}$

$$\lim_{s \rightarrow -1} \frac{(s+1)^3(2s^2 + 5s)}{(s+1)^3} = -3 \qquad \lim_{s \rightarrow -1} \frac{d}{ds} \left[\frac{(s+1)^3(2s^2 + 5s)}{(s+1)^3} \right] = 1$$

$$\frac{1}{2!} \lim_{s \rightarrow -1} \frac{d^2}{ds^2} \left[\frac{(s+1)^3(2s^2 + 5s)}{(s+1)^3} \right] = 2$$

$$\mathcal{L}^{-1} \left[\frac{2s^2 + 5s}{(s+1)^3} \right] = e^{-t}(2 + t - 3t^2)u(t)$$

Not Strictly Proper Laplace Transforms

Find the inverse LT of $F(s) = \frac{s^3 + 6s^2 + 12s + 8}{s^2 + 4s + 3}$

Convert to polynomial plus strictly proper rational function

Use polynomial division

$$\begin{aligned} F(s) &= s + 2 + \frac{s + 2}{s^2 + 4s + 3} \\ &= s + 2 + \frac{0.5}{s + 1} + \frac{0.5}{s + 3} \end{aligned}$$

Invert as normal

$$f(t) = \left[\delta'(t) + 2\delta(t) + 0.5e^{-t} + 0.5e^{-3t} \right] u(t)$$

System stability

Easy to determine using the transfer function $H(s)$

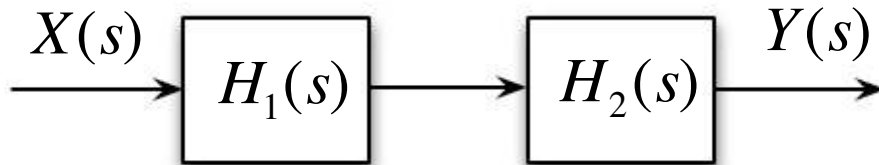
System is **BIBO stable** if

all the **poles** of the transfer function lie in the **open left half** of the **s plane**

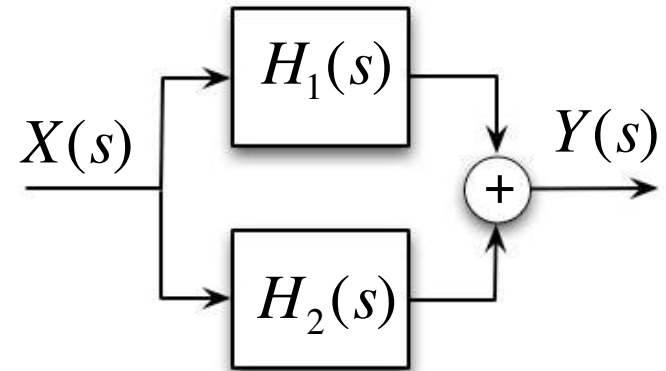
The system is called **marginally stable** if there are simple poles on the imaginary axis and no poles in the right half-plane.

Marginally stable is a particular case of BIBO unstable.

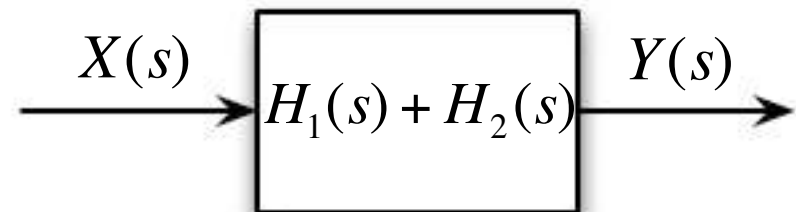
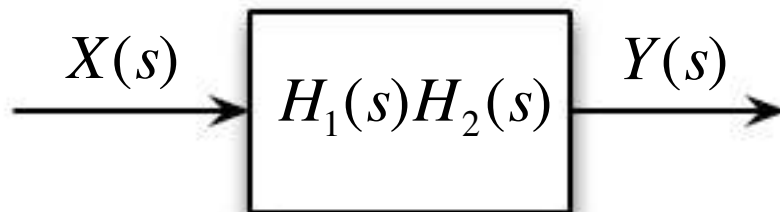
System interconnections



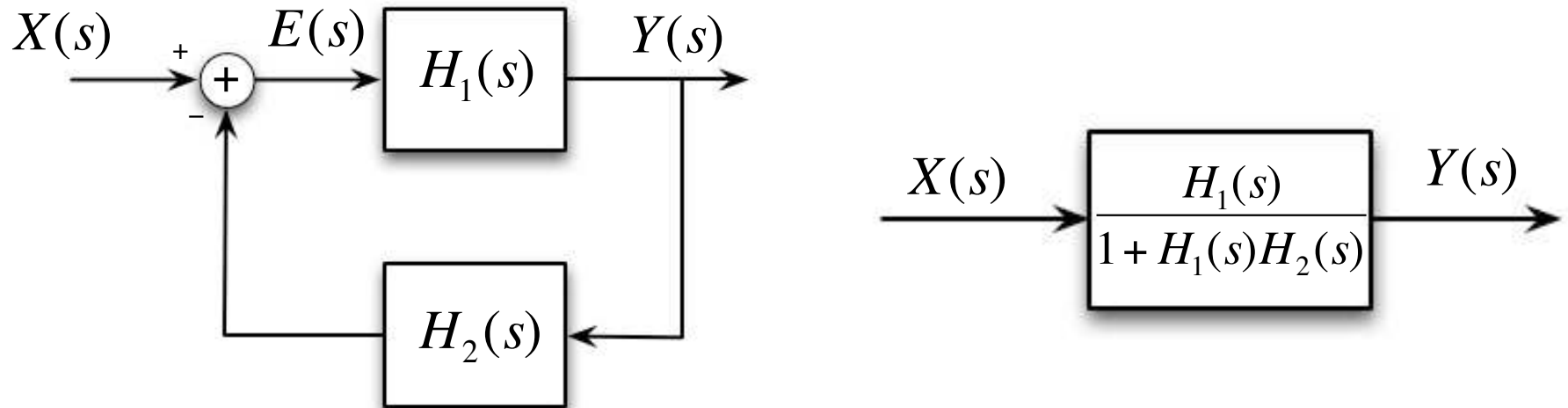
Cascade interconnection



Parallel interconnection



Feedback interconnection



$H_1(s)$ is the **plant** or original system we want to control

$H_2(s)$ is the **sensor/controller** selected to make closed-loop system have certain property (e.g., stability)

Example: $H_1(s) = \frac{1}{s}$, $H_2(s) = K$, $\frac{H_1(s)}{1 + H_1(s)H_2(s)} = \frac{1}{s + K}$

Much more about this in **MAE143B...**

Block diagrams - revisited

Multiple ways of drawing a system block diagram for strictly proper ($n \geq m$) transfer functions

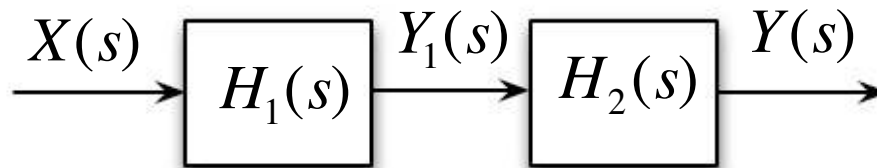
$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

(if $n > m$, then some of the b_i are zero)

- $H(s)$ as a **product**, $H_1(s)H_2(s)$

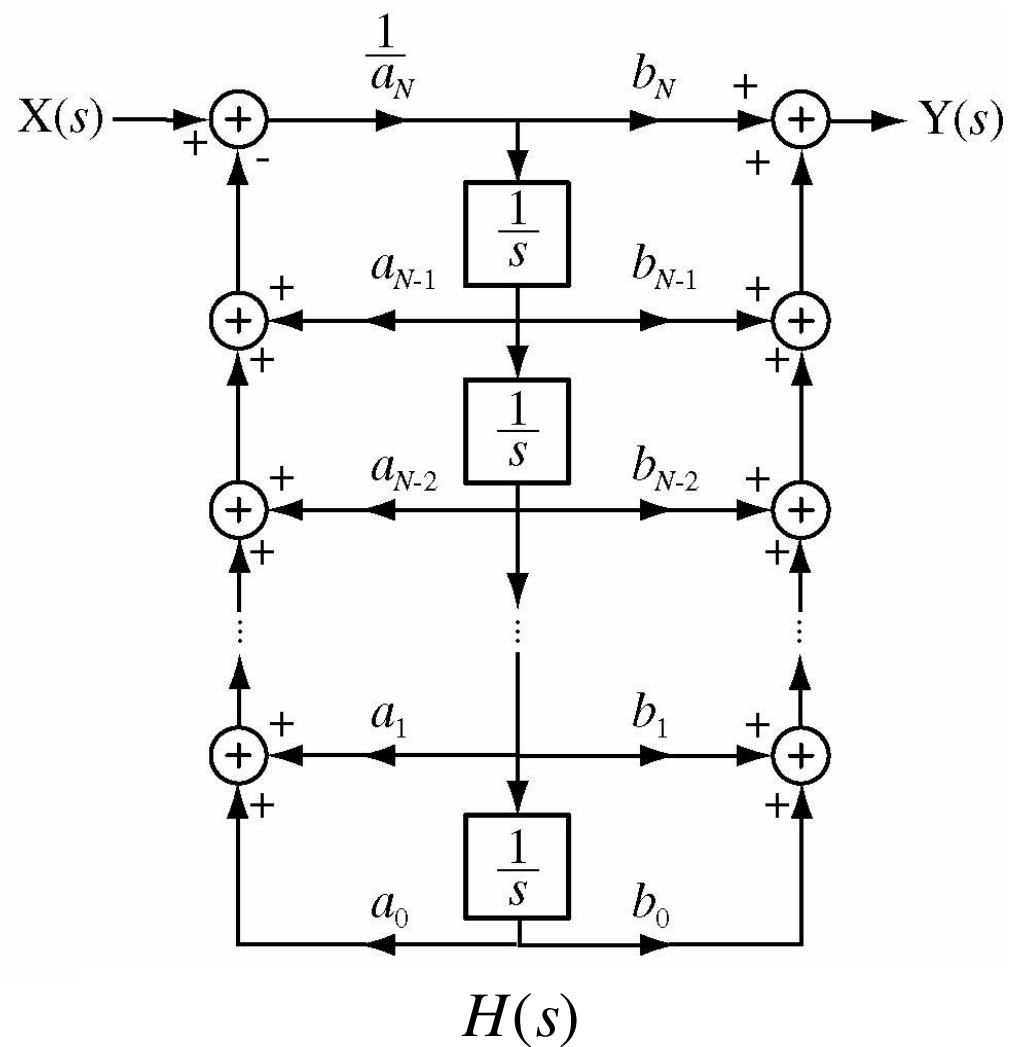
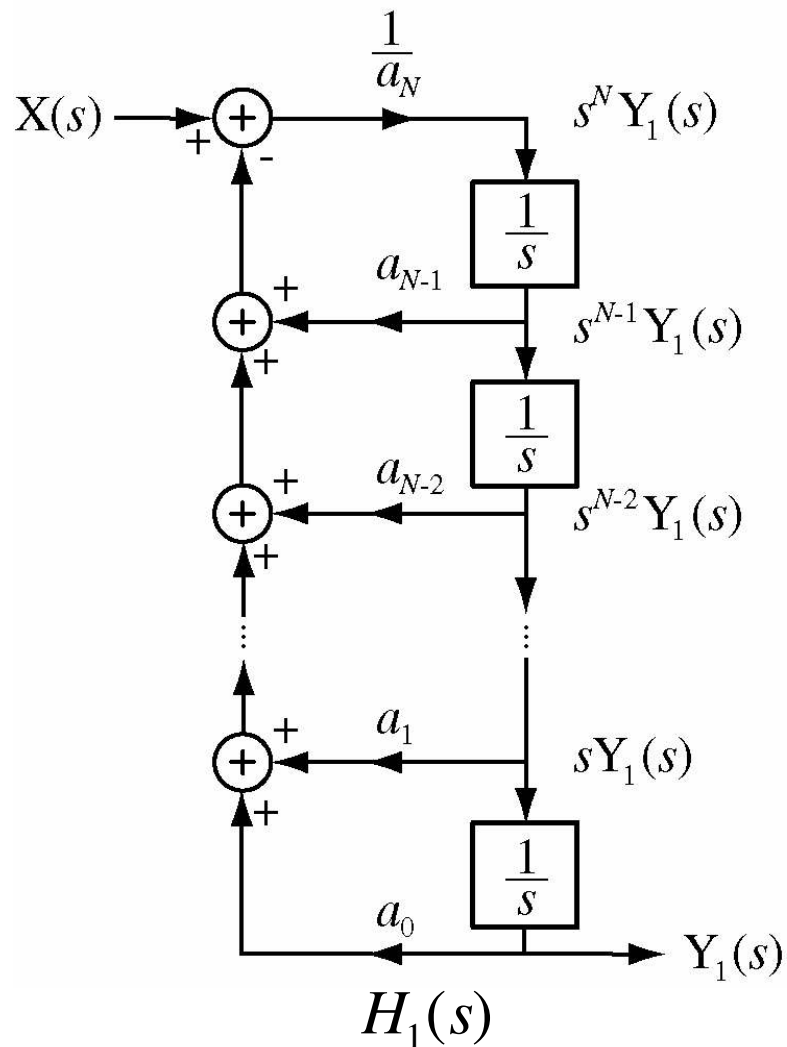
$$H_1(s) = \frac{Y_1(s)}{X(s)} = \frac{1}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$H_2(s) = \frac{Y(s)}{Y_1(s)} = b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0$$



Block diagrams - revisited

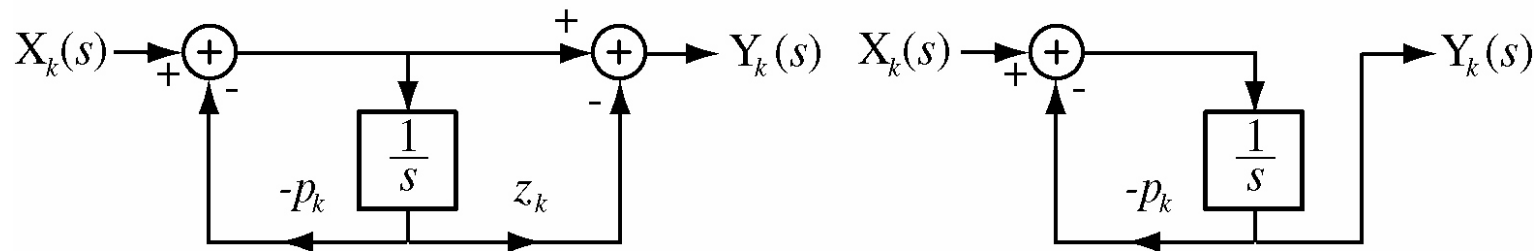
$$s^N Y_1(s) = \frac{1}{a_N} \left\{ X(s) - \left[a_{N-1} s^{N-1} Y_1(s) + \cdots + a_1 s Y_1(s) + a_0 Y_1(s) \right] \right\}$$



Block diagrams - revisited

- in **cascade** form

$$H(s) = A \frac{(s - z_1) \dots (s - z_m)}{(s - p_1) \dots (s - p_m)} \frac{1}{(s - p_{m+1})} \dots \frac{1}{(s - p_n)}$$



- in **parallel** form

$$H(s) = \frac{K_1}{(s - p_1)} + \dots + \frac{K_n}{(s - p_n)}$$

