Computing the Impulse Response

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Consider the problem of computing the impulse response $h(t)$ of a *Linear Time-Invariant* (LTI) System described by a linear *Ordinary Differential Equation* (ODE). That is, computing the solution to the linear ODE

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) = b_0 x^m(t) + \cdots + b_m x(t) \tag{1}$$

starting from *zero initial conditions* when the input is set to

$$x(t) = \delta(t). \tag{2}$$

1 A Simplified Problem

We start by computing the solution to a simplified problem. The reason for this will become clear later.

Let us start by computing the impulse response $h_z(t)$ to the auxiliary ODE

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) = z(t) \tag{3}$$

i.e., where $b_m = 1$, and $b_i = 0$ for all $i = 1, \ldots, m-1$.

Setting the input to

$$z(t) = \delta(t), \tag{4}$$

and integrating (3) on the left and on the right from $-\infty$ to t we obtain

$$y^{(n-1)}(t) + a_1 y^{(n-2)}(t) + \cdots + a_n \int_{-\infty}^{t} y(\tau)\, d\tau = \int_{-\infty}^{t} \delta(\tau)\, d\tau. \tag{5}$$

where we have used the fact that at $-\infty$ the value of $y(t)$ and all its derivatives is zero, due to the hypothesis of zero initial conditions.

Now note that solutions to equations (3) with (4) must be *impulse free*. Indeed, if $y(t)$ contains an impulse, then the left hand side of (3) will have a term $\delta^{(n)}(t)$ which will not be matched at the right hand side. Furthermore, because of the zero initial conditions

$$\int_{-\infty}^{t} y(\tau)\, d\tau = \int_{0}^{t} y(\tau)\, d\tau \tag{6}$$

and from (5)

$$y^{(n-1)}(t) + a_1 y^{(n-2)}(t) + \cdots + a_{n-1} y(t) = f_1(t)\, u(t) \tag{7}$$

where

$$f_1(t) := 1 - a_n \int_0^t y(\tau)\, d\tau. \tag{8}$$

Integrating (7) from $-\infty$ to t yields

$$y^{(n-2)}(t) + a_1 y^{(n-3)}(t) + \cdots + a_{n-2} y(t) = f_2(t)\, u(t) \tag{9}$$

where

$$f_2(t) := \int_0^t f_1(\tau)\, d\tau - a_{n-1} \int_0^t y(\tau)\, d\tau \tag{10}$$

One can then successively integrate (9) $n-2$ times in order to obtain

$$\begin{aligned} y^{(n-3)}(t) + \cdots + a_{n-4} y^{(1)}(t) + a_{n-3} y(t) &= f_3(t)\, u(t) \\ &\vdots \\ y^{(1)}(t) + a_1 y(t) &= f_{n-1}(t)\, u(t) \\ y(t) &= f_n(t)\, u(t) \end{aligned}$$

where $f_i(t)$, $i = 3, \ldots, n$ will consist of summation of integrals. The exact form of the f_i's is not of much importance here. What is important is that at $t = 0^+$ we have

$$f_1(0^+) = 1, \qquad f_i(0^+) = 0 \quad \text{for all } i = 2, \ldots, n. \tag{11}$$

Therefore

$$y^{(n-1)}(0^+) = 1, \qquad y^{(n-i)}(0^+) = 0 \quad \text{for all } i = 2, \ldots, n. \tag{12}$$

As $\delta(t) = 0$ for all $t > 0$, the solution $h_z(t) = y(t)$ for $t > 0$ can be computed by solving the homogeneous version of equation (3) subject to the initial conditions (12). The final form of the solution will therefore depend on the coefficients a_i's, which we illustrate with a simple example.

Example 1. *Problem: Compute the impulse response of*

$$y''(t) + ay'(t) = x(t).$$

Answer: Solutions to the homogeneous equation

$$y''(t) + ay'(t) = 0$$

are of the form $y(t) = ke^{\lambda t}$ where λ is one of the roots of the characteristic polynomial

$$\lambda^2 + a\lambda = 0.$$

That is

$$y(t) = k_1 + k_2 e^{-at}.$$

For the impulse response we set

$$y(0^+) = k1 + k_2 = 0, \quad y'(0^+) = -ak_2 = 1 \qquad \Longrightarrow \qquad k_1 = -k_2, \quad k_2 = -1/a.$$

Hence

$$h_z(t) = (1/a)(1 - e^{-at})u(t).$$

2 Back to the Original Problem

In this section, we use the solution of the simplified problem of Section 1 to solve our original problem (1). We divide the treatment in three cases, according to $m < n$, $m = n$, and $m > n$.

2.1 The Case $m < n$

We are now ready to consider the original ODE (1) under the assumption that $m < n$. In this case, one approach for computing the impulse response is to first compute the impulse response $h_z(t)$ to the auxiliary ODE (3) then use linearity to compute

$$h(t) = b_0 h_z^{(m)}(t) + \cdots + b_m h_z(t). \tag{13}$$

Note that for (13) to be valid we need $h(t)$ to be *impulse free*. Using an argument similar to the one used in the previous section one can conclude that this is precisely the case when $m < n$. For the same reason, the derivatives of $h_z(t)$ are to be evaluated only for $t > 0$. A simple example will clarify the use of the above formula.

Example 2. *Problem: Compute the impulse response of*

$$y''(t) + ay'(t) = 2x(t) - x'(t).$$

Solution: In this case $m = 1$ and $n = 2$ hence $m < n$ and the impulse response to the auxiliary ODE (3) is

$$h_z(t) = (1/a)(1 - e^{-at})u(t)$$

which we have computed before. Therefore

$$\begin{aligned} h(t) &= 2h_z(t) - h'_z(t) \\ &= [2(1/a)(1 - e^{-at}) - e^{-at}]u(t) \\ &= (2/a)u(t) - [(2 - a)/a]e^{-at}u(t). \end{aligned}$$

Note that $h'_z(t)$ is computed at $t > 0$ and therefore $h'_z(t) = e^{at}u(t) \neq (1/a)\delta(t) + e^{at}u(t)$, which is what you would get if $h'_z(t)$ were to be computed for all t!

2.2 The Case $m = n$

In this case, and indeed for $m \geq n$, the impulse response will contain at least one impulse. Because of this we cannot simply differentiate $h_z(t)$. Instead, we first rearrange

$$\begin{aligned} y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) = b_0[x^{(n)}(t) + a_1 x^{(n-1)}(t) + \cdots + a_n x(t)] \\ + (b_1 - b_0 a_1)x^{(n-1)}(t) + \cdots + (b_n - b_0 a_n)x(t). \end{aligned}$$

Note that this is the same as

$$\begin{aligned} \left(y^{(n)}(t) - b_0 x^{(n)}(t)\right) + a_1 \left(y^{(n-1)}(t) - b_0 x^{(n-1)}(t)\right) + \cdots + a_n \left(y(t) - b_0 x(t)\right) \\ = (b_1 - b_0 a_1)x^{(n-1)}(t) + \cdots + (b_n - b_0 a_n)x(t). \end{aligned}$$

Defining

$$g(t) := y(t) - b_0 x(t) \tag{14}$$

we can rewrite the above equation as the equivalent ODE

$$g^{(n)}(t) + a_1 g^{(n-1)}(t) + \cdots + a_n g(t) = (b_1 - b_0 a_1)x^{(n-1)}(t) + \cdots + (b_n - b_0 a_n)x(t) \tag{15}$$

which now has $m = n - 1 < n$. If we compute $h_g(t)$ using the method of the previous section we can then obtain the impulse response $h(t)$ by using linearity

$$h(t) = h_g(t) + b_0 \delta(t). \tag{16}$$

Note that the coefficients of the ODE (15) correspond to the Euclidean division of the polynomial $n(s)$ by the polynomial $d(s)$ defined as

$$n(s) := b_0 s^n + b_1 s^{n-1} + \cdots + b_n, \qquad d(s) := s^n + a_1 s^{n-1} + \cdots + a_n. \tag{17}$$

Indeed

$$n(s) = b_0\, d(s) + (b_1 - b_0 a_1)s^{n-1} + \cdots + (b_n - b_0 a_n). \tag{18}$$

Example 3. *Problem: Compute the impulse response of*

$$y''(t) + ay'(t) = x''(t).$$

Solution: Rearranging we obtain

$$g''(t) + ag'(t) = -ax'(t)$$

from where

$$h_g(t) = -ah'_z(t) = -ae^{-at}u(t)$$

and finally

$$h(t) = h_g(t) + \delta(t) = -ae^{-at}u(t) + \delta(t).$$

2.3 The Case $m > n$

The case $m > n$ is handled similarly to the case $m = n$. Indeed, let

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) = b_0 x^{(m)}(t) + b_1 x^{(m-1)}(t) + \cdots + b_m x(t) \tag{19}$$

have $b_0 \neq 0$. As before rearrange to obtain

$$\begin{aligned} y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) = b_0[x^{(m)}(t) + a_1 x^{(m-1)}(t) + \cdots + a_n x^{(m-n)}(t)] \\ (b_1 - b_0 a_1)x^{(m-1)}(t) + \cdots + (b_n - b_0 a_n)x^{(m-n)}(t) \\ + b_{n+1}x^{(m-n-1)}(t) + \cdots + b_m x(t). \end{aligned}$$

After defining

$$g_1(t) := y(t) - b_0 x^{(m-n)}(t) \tag{20}$$

we obtain the equivalent ODE

$$\begin{aligned} g_1^{(n)}(t) + a_1 g_1^{(n-1)}(t) + \cdots + a_n g_1(t) = (b_1 - b_0 a_1)x^{m-1}(t) + \cdots + (b_n - b_0 a_n)x(t) \\ + b_{n+1}x^{(m-n-1)}(t) + \cdots + b_m x(t). \end{aligned} \tag{21}$$

which now has the highest degree derivative of degree $m - 1$. One can therefore continue the process until the highest degree becomes less or equal to n, in which case the impulse response can be calculated using the previously discussed strategies.

As in the previous section, the resulting differential equation in terms of an appropriately defined variable $g_{m-n}(t)$ has coefficients that correspond to the Euclidean division of the polynomial $n(s)$ by the polynomial $d(s)$:

$$n(s) := b_0 s^m + b_1 s^{m-1} + \cdots + b_m, \qquad d(s) := s^n + a_1 s^{n-1} + \cdots + a_n. \tag{22}$$

Indeed if $k(s)$ and $r(s)$ are polynomials such that

$$n(s) = k(s)\, d(s) + r(s) \tag{23}$$

with $\deg(r) < n$ then

$$g_{m-n}(t) := y(t) - [k_0 x^{(m-n)}(t) + \cdots + k_{m-n} x(t)]$$

and the resulting differential equation will be of the form

$$g_{m-n}^{(n)}(t) + a_1 g_{m-n}^{(n-1)}(t) + \cdots + a_n g_{m-n}(t) = r_0 x^{(n-1)}(t) + \cdots + r_{n-1} x(t)$$

The corresponding impulse response is of the form

$$h(t) = h_g(t) + k_0 \delta^{(m-n)}(t) + \cdots + k_{m-n} \delta(t). \tag{24}$$

As we will see later, the above procedure can be easily proved formally using Laplace transforms, which will bring a meaningful interpretation for the polynomials $n(s)$ and $d(s)$.

Example 4. *Problem: Compute the impulse response of*

$$y''(t) + a y'(t) = x'''(t).$$

Solution: The associated polynomials are

$$n(s) = s^3, \qquad d(s) = s^2 + as.$$

Therefore

$$n(s) = k(s)\, d(s) + r(s), \qquad k(s) = s - a, \qquad r(s) = a^2 s$$

Rearranging we obtain

$$g''(t) + a g'(t) = a^2 x'(t)$$

from where

$$h_g(t) = a^2 h_z'(t) = a^2 e^{-at} u(t)$$

and finally

$$h(t) = h_g(t) + \delta'(t) - a\delta = a^2 e^{-at} u(t) + \delta'(t) - a\delta(t).$$
