MAE143A - Signals and Systems - Fall 2021
Final, December 7th

Instructions

1. This exam is open book. You may use whatever written or printed materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.
2. You have 120 minutes.
3. Write your answers on your "Blue Book".
4. Do not forget to write your name, student number, and instructor.
5. This exam has 4 questions, for a total of 40 points and 3 bonus points.

1. Signals and Transforms

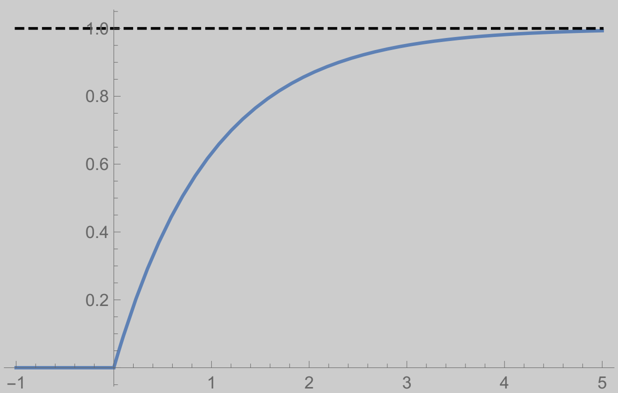
   Consider the continuous-time signal

$$x(t) = (1 - e^{-t})\, 1(t)$$

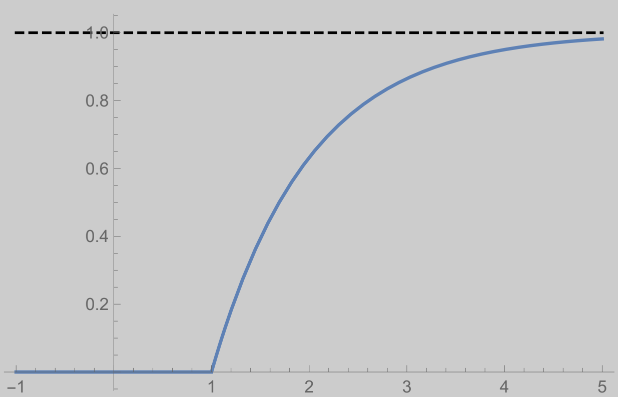
   (a) (3 points) Sketch the signals $x(t)$, $x(t-1)$, $x(1-t)$.

Solution:

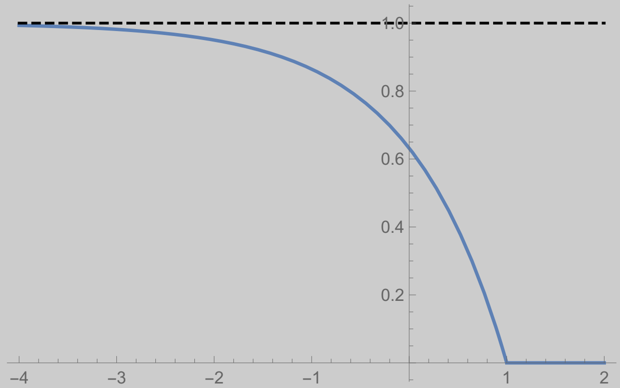
Sketches should look like:



(+1 point)



(+1 point)



(+1 point)

(b) (2 points) Calculate the generalized derivatives $\dot{x}(t)$ and $\ddot{x}(t)$.

Solution:

The generalized first derivative is

$$\dot{x}(t) = e^{-t}\,1(t) \qquad (+1 \text{ point})$$

and the generalized second derivative is

$$\ddot{x}(t) = \delta(t) - e^{-t}\,1(t) \qquad (+1 \text{ point})$$

(c) (2 points) What is the Fourier transform of $x(t)$?

Solution:

From a table

$$\mathcal{F}\{x(t)\} = \mathcal{F}\{1(t)\} - \mathcal{F}\{e^{-t}1(t)\} = \frac{1}{j\omega} + \pi\delta(\omega) - \frac{1}{j\omega + 1}. \qquad (+2 \text{ points})$$

(d) (2 points) What is the Laplace transform of $x(t)$?

Hint: do not forget the region of convergence!

Solution:

From a table

$$\mathcal{L}\{x(t)\} = \mathcal{L}\{1(t)\} - \mathcal{L}\{e^{-t}1(t)\} = \frac{1}{s} - \frac{1}{s+1}, \quad \mathrm{Re}\{s\} > 0. \qquad (+2 \text{ points})$$

(e) (2 points) Let $X(s) = \mathcal{L}\{x(t)\}$. Is $X(j\omega) = \mathcal{F}\{x(t)\}$? Explain.

Solution:

No, as can be seen above. (+1 point)

Because the imaginary axis is not contained in the region of convergence of the Laplace transform. (+1 point)

(f) (1 point) Calculate $\lim_{t \to 0^+} x(t)$. Repeat the calculation using the initial value property of the Laplace transform.

Solution:

Evaluate

$$\lim_{t\to 0^+} x(t) = 0, \qquad \lim_{s\to\infty} sX(s) = 1 - \lim_{s\to\infty} \frac{s}{s+1} = 0. \qquad (+1 \text{ point})$$

(g) (1 point) Calculate $\lim_{t\to\infty} x(t)$. Repeat the calculation using the final value property of the Laplace transform.

Solution:

Evaluate

$$\lim_{t\to\infty} x(t) = 1, \qquad \lim_{s\to 0^+} sX(s) = 1 - \lim_{s\to 0^+} \frac{s}{s+1} = 1. \qquad (+1 \text{ point})$$

(h) (3 points) Calculate the transfer-function of the causal LTI system for which $y(t) = x(t)$ is the zero-state response to the input $u(t) = 1(t)$. Is the system asymptotically stable?

Solution:

The transfer-function is

$$G(s) = \frac{Y(s)}{U(s)} \qquad (+1 \text{ point})$$

Calculating

$$G(s) = \frac{\frac{1}{s} - \frac{1}{s+1}}{\frac{1}{s}} = \frac{\frac{1}{s(s+1)}}{\frac{1}{s}} = \frac{1}{s+1}, \qquad \mathrm{Re}\{s\} > -1 \qquad (+1 \text{ point})$$

which is asymptotically stable because it does not have any poles with positive real part. (+1 point)

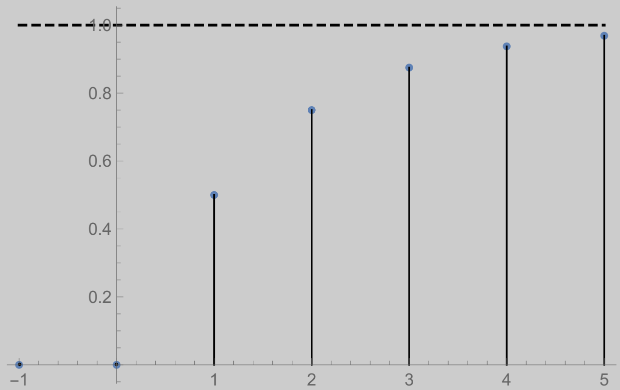
(i) (1 point) Sketch the signal $x[k] = x(hk)$, $h = \ln(2)$.

Solution:

Note that

$$x[k] = (1 - e^{-\ln(2)k})1(k) = (1 - 2^{-k})1(k).$$

A sketch should look like:



(+1 point)

2. Discrete-time System

   A causal discrete-time system is modeled by the difference equation

$$y[k] = \frac{1}{2}y[k-1] + \frac{1}{2}u[k]$$

(a) (2 points) Calculate the system's transfer-function $G(z)$ and its region of convergence.

Solution: By inspection from the difference equation

$$G(z) = \frac{1/2}{1-(1/2)z^{-1}} = \frac{z}{2z-1}. \qquad (+1 \text{ point})$$

Its region of convergence is $|z| > 1/2$. (+1 point)

(b) (1 point) Calculate the transfer-function's poles and zeros.

Solution:
Zero at

$$z = 0 \qquad (+1/2 \text{ point})$$

and pole at

$$z = 1/2. \qquad (+1/2 \text{ point})$$

(c) (1 point) Is the system asymptotically stable? Explain.

Solution: Yes because the pole $z = 1/2$ is inside the unit circle. (+1 point)

(d) (2 points) Calculate the first 4 non-zero terms of the system's impulse response, that is $g[k]$, $k = 0, \cdots, 3$. Generalize the solution to all k.

Hint: no need to justify the general solution.

Solution:

$$
\begin{aligned}
y[0] &= \frac{1}{2}y[-1] + \frac{1}{2}\delta[0] = \frac{1}{2}, && \text{(+1 point)}\\
y[1] &= \frac{1}{2}y[0] + \frac{1}{2}\delta[1] = \frac{1}{4},\\
y[2] &= \frac{1}{2}y[1] + \frac{1}{2}\delta[2] = \frac{1}{8},\\
y[3] &= \frac{1}{2}y[2] + \frac{1}{2}\delta[3] = \frac{1}{16}.
\end{aligned}
$$

The general solution is

$$y[k] = 2^{-(k+1)}1[k] \qquad \text{(+1 point)}$$

3. Continuous-time System

Consider the causal LTI system described by the transfer function:

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = u(t)$$

(a) (2 points) Calculate the system's transfer-function $G(s)$ and its region of convergence.

Solution:

By inspection

$$G(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1. \qquad \text{(+2 points)}$$

(b) (1 point) Calculate the transfer-function's poles and zeros.

Solution:

No zeros, poles at:

$$s = -1, \quad s = -2. \qquad \text{(+1 point)}$$

(c) (1 point) Is the system asymptotically stable? Explain.

Solution:

Yes, both poles are negative.

(d) (3 points) Calculate the system's impulse response.

Solution:

Expanding in partial fractions

$$G(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2} \qquad \text{(+2 points)}$$

from which

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = (e^{-t} - e^{-2t})1(t). \qquad (+1 \text{ point})$$

(e) (4 points) Calculate the system's response to the input $u(t) = 1(t)$ given the initial conditions $y(0^+) = 1$, $\dot{y}(0^+) = 0$. What is the value of the initial state?

Solution:

Transform of the input is:

$$U(s) = \frac{1}{s} \qquad (+1/2 \text{ point})$$

and the response is

$$Y(s) = G(s)U(s) + \frac{x_2(0^+) + (s+3)x_1(0^+)}{(s+1)(s+2)} \qquad (+1/2 \text{ point})$$

Because the system has no zeros, the initial state is equal to the initial condition, that is

$$x_1(0^+) = y(0^+) = 1, \qquad x_2(0^+) = \dot{y}(0^+) = 0 \qquad (+1 \text{ point})$$

Expanding in partial fractions

$$\begin{aligned} Y(s) &= \frac{1}{s(s+1)(s+2)} + \frac{s+3}{(s+1)(s+2)} \\ &= \frac{1/2}{s} + \frac{2-1}{s+1} + \frac{1/2-1}{s+2} \\ &= \frac{1/2}{s} + \frac{1}{s+1} - \frac{1/2}{s+2} \qquad (+1 \text{ point}) \end{aligned}$$

and

$$y(t) = \left(\frac{1}{2} + e^{-t} - \frac{1}{2}e^{-2t}\right)1(t) \qquad (+1 \text{ point})$$

NOTE TO GRADER: an alternative formulation is

$$Y(s) = G(s)U(s) + \frac{x_2(0^+) + sx_1(0^+)}{(s+1)(s+2)}$$

with

$$x_1(0^+) = y(0^+) = 1, \qquad x_2(0^+) = \dot{y}(0^+) + 3y(0^+) = 3$$

which leads to

$$Y(s) = \frac{1}{s(s+1)(s+2)} + \frac{s+3}{(s+1)(s+2)}$$

as well.

(f) (4 points) Calculate $|G(j\omega)|$ and $\angle G(j\omega)$. Evaluate $|G(j\omega)|$ for $\omega = \{0, 1, 2, \infty\}$, and sketch $|G(j\omega)|$. What type of filter response is displayed by this system?

Solution:

Evaluating

$$|G(j\omega)| = \frac{1}{|j\omega + 1||j\omega + 2|} = \frac{1}{\sqrt{(\omega^2 + 1)}\sqrt{(\omega^2 + 4)}}, \qquad (+1 \text{ point})$$

and

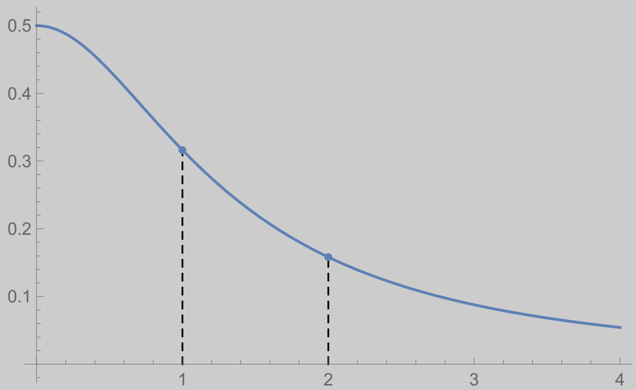
$$\angle G(j\omega) = -\angle(j\omega + 1) - \angle(j\omega + 2) = -\tan^{-1}\omega - \tan^{-1}\omega/2 \qquad (+1 \text{ point})$$

Values of the magnitude are

$$|G(j0)| = \frac{1}{2}, \qquad |G(j1)| = \frac{1}{\sqrt{2}\sqrt{5}} = \frac{\sqrt{10}}{10} \approx 0.31, \qquad (+1 \text{ point})$$

$$|G(j\infty)| = 0, \qquad |G(j2)| = \frac{1}{\sqrt{5}\sqrt{8}} = \frac{\sqrt{10}}{20} \approx 0.16,$$

Sketch should look like:



(+1/2 point)

which displays a low-pass behavior. (+1/2 point)

(g) (2 points) Use the frequency response to calculate the steady-state component of the response to the input $u(t) = \sin(t)$. Hint: $\sin(t) = \cos(t - \pi/2)$.

Solution:

Because the system is asymptotically stable

$$y(t) = |G(j1)| \cos(t - \pi/2 + \angle G(j1)) \qquad (+1 \text{ point})$$

$|G(j1)| = \sqrt{10}/10 \approx 0.31$ from the previous question and

$$\angle G(j1) = -\tan(1) - \tan(1/2) \approx -1.25 \text{ rad} \approx -72^\circ \qquad (+1/2 \text{ point})$$

and

$$y(t) \approx 0.13 \cos(t - 2.82) \approx 0.13 \cos(t - 161^\circ) \qquad (+1/2 \text{ point})$$

4. Fourier Series and Filtering

Some students used a spectrum analyzer to measure the Fourier series components of the output $\hat{Y}[n]$ of an asymptotically stable causal LTI system with transfer-function $G(s)$ driven by the periodic signal with Fourier series components $\hat{X}[n]$.

(a) (2 points (bonus)) The magnitude of the output coefficients were the same as the magnitude of the input coefficients but the phase of the odd components were shifted by 180°, that is $|\hat{Y}[n]| = |\hat{X}[n]|$, for all n, $\angle\hat{Y}[n] = \angle\hat{X}[n]$ if n is even, and $\angle\hat{Y}[n] = \angle\hat{X}[n] - \pi$ if n is odd. Find a transfer-function that can explain this behavior. Hint: try a transfer-function that is not rational.

Solution:

The system response is given by

$$\hat{Y}[n] = G(jn\omega_0)\hat{X}[n].$$

If $G(s) = e^{-s\tau}$, which corresponds to a pure delay, then

$$G(jn\omega_0) = e^{-jn\omega_0\tau}$$

Setting $\tau = \pi/\omega_0$

$$G(jn\omega_0) = e^{-jn\pi} = \begin{cases} 1, & n \text{ is even,} \\ e^{-j\pi}, & n \text{ is odd,} \end{cases}$$

which explains the observed behavior. (+2 bonus points)

(b) (1 point (bonus)) The students observed that when the input signal was added to the output signal the magnitude of the even components were double and the magnitude of the odd components were zero. Explain this phenomenon.

Hint: you can answer this item even if you did not answer item (a).

Solution:

Note that $Y[n] = e^{-j\pi}X[n] = -X[n]$ if n is odd so that

$$\hat{X}[n] + \hat{Y}[n] = \begin{cases} \hat{X}[n] + \hat{X}[n] = 2\hat{X}[n], & n \text{ is even,} \\ \hat{X}[n] - \hat{X}[n] = 0, & n \text{ is odd.} \end{cases}$$

(+1 bonus point)