MAE143A - Signals and Systems - Fall 2021
Midterm #1, October 21st

Instructions

1. This exam is open book. You may use whatever written or printed materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.
2. You have 70 minutes.
3. Write your answers on your "Blue Book".
4. Do not forget to write your name, student number, and instructor.
5. This exam has 5 questions, for a total of 45 points.

1. **Signals**

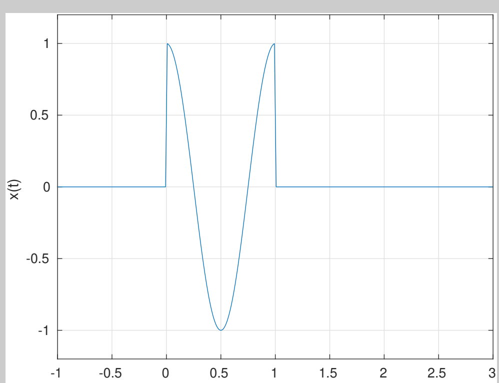
   Consider the signal

$$x(t) = \cos(2\pi t)\ (1(t) - 1(t-1))\,,$$

   and answer the following questions:

   (a) (3 points) Sketch the signal $x(t)$;

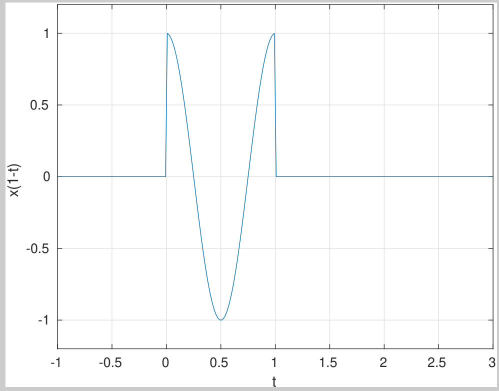
   Solution: Solution is:



   (+3 points)

   (b) (3 points) Sketch the signal $x(1-t)$;

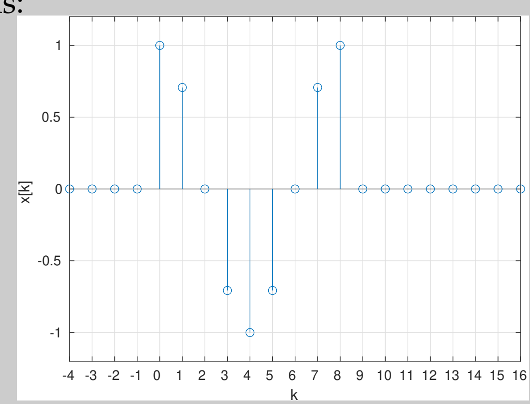
   Solution: Solution is:



(+3 points)

(c) (3 points) Sketch the signal $x[k] = x(kT)$, $T = 1/8$;

Solution: Solution is:



(+3 points)

NOTE TO GRADERS: The value at $k = 0$ ($t = 0$) and $k = 8$ ($t = 1$) can be different depending on how the student defined the value of $1(0)$. For example, if $1(0) = 1/2$, then $x[0] = x[k] = 1/2$.

NOTE: Make sure your sketches contain all information needed to clearly identify the signals.

2. **Continuous-Time Impulse Response**

Consider a continuous-time LTI system with impulse response

$$g(t) = e^{-t}\,1(t),$$

and answer the following questions:

(a) (2 points) Is this system causal? Explain.

Solution: Yes, because $g(t) = 0$ for $t < 0$. (+2 points)

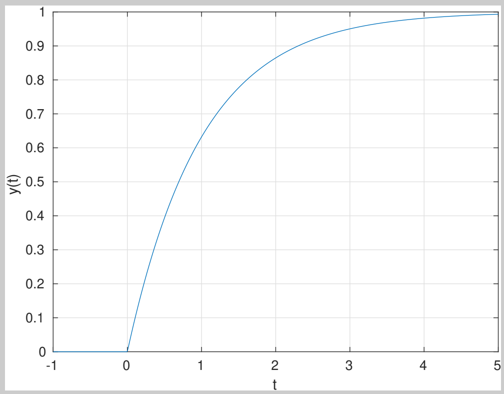
(b) (4 points) Use convolution to calculate the response to a unit step, $u(t) = 1(t)$. Sketch the response.

Solution: The response is the signal

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} g(\tau)\, 1(t-\tau)\, d\tau \\ &= \begin{cases} 0, & t < 0 \\ \int_0^t e^{-t} d\tau = 1 - e^{-t}, & t \geq 0 \end{cases} \\ &= (1 - e^{-t})\, 1(t). \end{aligned}$$

(+2 points)

Sketch is:



(+2 points)

(c) (3 points) Is the system BIBO stable? Show your work.

Solution: Because $g(t) \geq 0$

$$\int_0^{\infty} |g(\tau)|\, d\tau = \lim_{t\to\infty} \int_0^t g(\tau)\, d\tau = \lim_{t\to\infty} \int_0^t e^{-\tau}\, d\tau$$

which you already calculated in part (b). (+1 point)

That is

$$\int_0^{\infty} |g(\tau)|\, d\tau = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} (1 - e^{-t}) = 1$$

(+1 point)

Since this is bounded the system is BIBO stable. (+1 point)

3. **ODE**

Consider a causal LTI system modeled by the ODE

$$2\,\dot{y}(t) + y(t) = u(t)$$

and answer the following questions:

(a) (3 points) Calculate the system's transfer-function and its region of convergence.

Solution: The transfer-function can be calculated by inspection from the coefficients of the ODE as

$$G(s) = \frac{1}{2s+1}. \qquad \text{(+1 point)}$$

Because the system is causal its region of convergence is $\mathrm{Re}\{s\} > -1/2$ because $s = -1/2$ is the pole of $G(s)$. (+2 points)

(b) (2 points) Is this system asymptotically stable? Explain.

Solution: The pole of the transfer-function is $s = -1/2$ so that $G(s)$ converges for all $\mathrm{Re}\{s\} > -1/2$ which includes the imaginary axis so the system is asymptotically stable.

(+2 points)

(c) (4 points) Use the transfer-function from part (a) to calculate the response of the system to the input $u(t) = \cos(t)$. Explain the assumptions needed in your approach.

Solution: From the frequency response formula

$$y(t) = |G(j\omega)|\cos(\omega t + \angle G(j\omega)), \qquad G(j\omega) = \frac{1}{2j\omega + 1}. \qquad \text{(+1 point)}$$

Substituting $\omega = 1$

$$|G(j1)| = \frac{1}{|2j+1|} = \frac{1}{\sqrt{5}}, \qquad G(j1) = -\tan^{-1}(2/1) \qquad \text{(+1 point)}$$

so that the response is

$$y(t) = \frac{\sqrt{5}}{5}\cos(t - \tan^{-1}(2)) \qquad \text{(+1 point)}$$

For using the frequency response formula the system needs to be asymptotically stable, which it is. See part (b). (+1 point)

4. **Difference Equation**

Consider a causal discrete-time LTI system described by the difference equation

$$y[k+1] - y[k] = u[k+1] + u[k]$$

and answer the following questions:

(a) (3 points) Calculate the system's transfer-function and its region of convergence.

Solution: The transfer-function can be calculated by inspection from the coefficients of the difference equation as

$$G(z) = \frac{z+1}{z-1}. \qquad \text{(+1 point)}$$

Because the system is causal its region of convergence is $|z| > 1$ because $z = 1$ is the pole of $G(z)$. (+2 points)

(b) (2 points) Is this system asymptotically stable? Explain.

Solution: The pole of the transfer-function is $z = 1$ so that $G(z)$ converges for all $|z| > 1$ which does not include the unit circle so the system is not asymptotically stable. (+2 points)

(c) (4 points) Calculate the first 5 terms of the system's impulse response.

Solution: Let

$$y[k] = y[k-1] + u[k] + u[k-1],$$

set $u[k] = \delta[k]$, $g[k] = 0$, $k < 0$ and evaluate

$$\begin{aligned}
g[0] &= g[-1] + u[0] + u[-1] = 1, \\
g[1] &= g[0] + u[1] + u[0] = 1 + 1 = 2, \\
g[2] &= g[1] + u[2] + u[1] = 2 = 2, \\
g[3] &= g[2] + u[3] + u[2] = 2, \\
g[4] &= g[3] + u[4] + u[3] = 2.
\end{aligned}$$

(+4 points)

5. **Rational Transfer-Function**

Consider the following transfer-function

$$G(s) = \frac{s}{s^2 + 3s + 2}$$

which is associated with a causal LTI continuous-time system and answer the following questions:

(a) (3 points) What is the transfer-function's region of convergence?

Solution: Factor the denominator

$$G(s) = \frac{s}{s^2 + 3s + 2} = \frac{s}{(s+1)(s+2)} \qquad \text{(+1 point)}$$

to see that the poles are $s = -1$ and $s = -2$. (+1 point)
Because the system is causal the region on convergence should be $\mathrm{Re}\{s\} > -1$. (+1 point)

(b) (2 points) Is the system asymptotically stable?

Solution: The poles of the transfer-function are $s = -1$ and $s = -2$ so that $G(s)$ converges for all $\mathrm{Re}\{s\} > -1$ which includes the imaginary axis so the system is asymptotically stable. (+2 points)

(c) (4 points) Expand the function in partial fractions.

Solution:

$$G(s) = \frac{k_1}{s+1} + \frac{k_2}{s+2} \qquad \text{(+1 point)}$$

in which

$$\begin{aligned} k_1 &= \lim_{s \to -1} (s+1)G(s) = \lim_{s \to -1} \frac{s}{s+2} = -1, \\ k_2 &= \lim_{s \to -2} (s+2)G(s) = \lim_{s \to -2} \frac{s}{s+1} = -2/-1 = 2. \end{aligned} \qquad \text{(+2 points)}$$

so that

$$G(s) = \frac{2}{s+2} - \frac{1}{s+1}. \qquad \text{(+1 point)}$$