MAE143A - Signals and Systems - Fall 2021
Midterm 2, November 18th

Instructions

1. This exam is open book. You may use whatever written or printed materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.
2. You have 70 minutes.
3. Write your answers on your "Blue Book".
4. Do not forget to write your name, student number, and instructor.
5. This exam has 4 questions, for a total of 33 points and 6 bonus points.

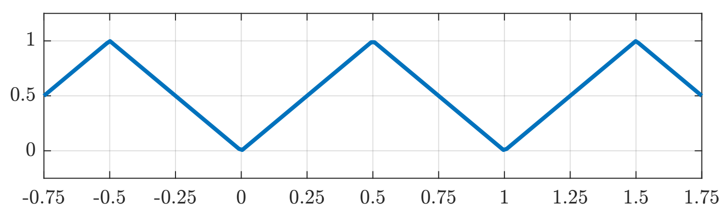

Figure 1: Figure for Question 1

1. **Fourier Series**

   Consider the periodic signal shown in Fig. 1 and answer the following questions:

   (a) (4 points) Show that the coefficients of its Fourier series are given by

   $$\hat{X}[n] = \begin{cases} \frac{1}{2}, & n = 0, \\ 0, & n \text{ even}, \\ \frac{-2}{n^2\pi^2}, & n \text{ odd} \neq 0. \end{cases}$$

   (b) (2 points) Calculate the Fourier transform of the signal.

   (c) (4 points) Calculate the following quantities

   $$\int_{T_0} x(t)\,dt, \qquad \int_{T_0} |x(t)|\,dt, \qquad \int_{T_0} |x(t)|^2\,dt.$$

   Hint: you have already calculated some of those quantities!

   (d) (2 points (bonus)) Show that

   $$\sum_{n=-\infty}^{\infty} \frac{4}{(2n+1)^4\pi^4} = \frac{1}{12}.$$

2. **Continuous-time System**

   A causal continuous-time linear time-invariant system is described by the ODE

   $$\dot{y} + y = \dot{u}.$$

   (a) (2 points) Calculate the system's transfer-function, $G(s)$, and its region of convergence.

   (b) (1 point) Is the system asymptotically stable? Explain.

   (c) (2 points) Calculate its impulse response, $g(t)$. *Hint: invert a transform!*

   (d) (1 point) Without performing any calculations, explain whether or not $G(j\omega) = \mathcal{F}\{g(t)\}$?

   (e) (4 points) Calculate the system's frequency response $G(j\omega)$ and sketch its magnitude and phase. What kind of filter response does this system display?

3. **System Response**

   (a) (4 points) Use either the Fourier or the Laplace transform to calculate the output response, $y(t)$, of the system from Question 2 to the input $u(t) = \mathrm{rect}(t - 1/2)$. Sketch the response.

   (b) (2 points) Calculate the coefficients of the Fourier series of the response of the system from Question 2 to the input from Question 1 with period $T_0 = 2\pi/10$.

   (c) (2 points) Calculate and sketch the magnitude of the first 4 nonzero coefficients of the Fourier series of the input and the magnitude of the first 4 nonzero coefficients of the Fourier series of the output.

   (d) (2 points) Based on your answers so far, are you able to approximately sketch the output signal? *Hint: compare the coefficients of the input with the coefficients of the output.*

4. **Filter Transformations**

   Answer the following questions.

   (a) (4 points (bonus)) Let $G(s)$ be a proper asymptotically stable rational transfer-function of order n for which $|G(0)| > 0$ and $|G(j)|^2 = 1/2$. Show that

   $$\tilde{G}(s) = G(\omega_c/s), \quad \omega_c > 0$$

   is such that $\tilde{G}(s)$ is a proper asymptotically stable rational transfer-function of order n and that

   $$|\tilde{G}(0)| = \lim_{\omega\to\infty} |G(j\omega)|^2, \qquad |\tilde{G}(j\omega_c)|^2 = |G(j)|^2 = 1/2, \qquad \lim_{\omega\to\infty} |\tilde{G}(j\omega)| = |G(0)|.$$

   Furthermore, if the degree of the denominator of $G(s)$ is n and the degree of the numerator is $m \leq n$ then $\tilde{G}(s)$ has $n - m$ zeros at zero.

   Hint: use a generic function of the form $G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$.

   (b) (1 point) Let

   $$G(s) = \frac{1}{s+1}, \quad \mathrm{Re}\{s\} > -1$$

   and calculate

   $$|G(j0)|, \qquad |G(j1)|^2, \qquad \lim_{\omega\to\infty} |G(j\omega)|.$$

   (c) (2 points) Apply the transformation from part (a) to the transfer-function in part (b) to calculate $\tilde{G}(s)$ and

   $$|\tilde{G}(j0)|, \qquad |\tilde{G}(j\omega_c)|^2, \qquad \lim_{\omega\to\infty} |G(j\omega)|.$$