MAE143A - Signals and Systems - Fall 2021
Midterm 2, November 18th

Instructions

1. This exam is open book. You may use whatever written or printed materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities.
2. You have 70 minutes.
3. Write your answers on your "Blue Book".
4. Do not forget to write your name, student number, and instructor.
5. This exam has 4 questions, for a total of 33 points and 6 bonus points.

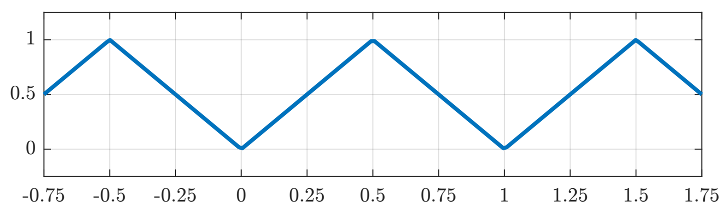


Figure 1: Figure for Question 1

1. **Fourier Series**

   Consider the periodic signal shown in Fig. 1 and answer the following questions:

   (a) (4 points) Show that the coefficients of its Fourier series are given by

$$\hat{X}[n] = \begin{cases} \dfrac{1}{2}, & n = 0, \\ 0, & n \text{ even}, \\ \dfrac{-2}{n^2\pi^2}, & n \text{ odd} \neq 0. \end{cases}$$

Solution: Note that $T_0 = 1$ and $\omega_0 = 2\pi$ so that

$$\begin{aligned}\hat{X}[n] &= \frac{1}{T_0}\int_{T_0} x(t)\, e^{-j2\pi nt}\, dt \\ &= \int_0^{1/2} 2te^{-j2\pi nt}\, dt + \int_{1/2}^{1} (2-2t)e^{-j2\pi nt}\, dt \\ &= 2\int_{1/2}^{1} e^{-j2\pi nt}\, dt + 2\int_0^{1/2} te^{-j2\pi nt}\, dt - 2\int_{1/2}^{1} te^{-j2\pi nt}\, dt. \qquad \text{(+2 points)}\end{aligned}$$

NOTE TO GRADER: Students get two points if they correctly identify the components in the integral.

A key integral is

$$\begin{aligned}
\int_a^b te^{-j2\pi nt}\,dt &= \left.\frac{te^{-j2\pi nt}}{-j2\pi n}\right|_{t=a}^{b} - \int_a^b \frac{e^{-j2\pi nt}}{-j2\pi n}\,dt \\
&= \left.\frac{te^{-j2\pi nt}}{-j2\pi n}\right|_{t=a}^{b} - \left.\frac{e^{-j2\pi nt}}{4\pi^2 n^2}\right|_{t=a}^{b} \\
&= \left.\frac{(1+j2\pi nt)e^{-j2\pi nt}}{4\pi^2 n^2}\right|_{t=a}^{b}.
\end{aligned}$$

Substituting back

$$\begin{aligned}
\frac{1}{2}\hat{X}[n] &= \frac{e^{-j2\pi n} - e^{-j\pi n}}{-j2\pi n} + \\
&\qquad \frac{(1+j\pi n)e^{-j\pi n}}{4\pi^2 n^2} - \frac{1}{4\pi^2 n^2} - \frac{(1+j2\pi n)e^{-j2\pi n}}{4\pi^2 n^2} + \frac{(1+j\pi n)e^{-j\pi n}}{4\pi^2 n^2} \\
&= \frac{j2\pi n(1-(-1)^n)}{4\pi^2 n^2} + \frac{(2+j2\pi n)(-1)^n}{4\pi^2 n^2} - \frac{(1+j2\pi n)}{4\pi^2 n^2} - \frac{1}{4\pi^2 n^2} \\
&= \frac{j2\pi n - 2 - j2\pi n + (-1)^n(2 - j2\pi n + j2\pi n)}{4\pi^2 n^2} \\
&= \frac{-1+1(-1)^n}{2\pi^2 n^2} \qquad \text{(+1 point)}
\end{aligned}$$

which is equal to the expression given.
Also

$$\begin{aligned}
\hat{X}[0] &= \frac{1}{T_0}\int_{T_0} x(t)\,dt \\
&= \int_0^{1/2} 2t\,dt + \int_{1/2}^{1} (2-2t)\,dt \\
&= 2\frac{1}{4} = \frac{1}{2}. \qquad \text{(+1 point)}
\end{aligned}$$

(b) (2 points) Calculate the Fourier transform of the signal.

Solution: The Fourier transform of the signal is

$$X(j\omega) = \sum_{n=-\infty}^{\infty} 2\pi\hat{X}[n]\delta(\omega - n\omega_0), \quad \omega_0 = \frac{2\pi}{T_0} = 2\pi. \qquad \text{(+2 points)}$$

(c) (4 points) Calculate the following quantities

$$\int_{T_0} x(t)\,dt, \qquad \int_{T_0} |x(t)|\,dt, \qquad \int_{T_0} |x(t)|^2\,dt.$$

Hint: you have already calculated some of those quantities!

Solution: Because the signal is non-negative and $T_0 = 1$

$$\int_{T_0} x(t)\, dt = \int_{T_0} |x(t)|\, dt = \frac{1}{T_0}\int_{T_0} x(t)\, dt = \hat{X}[0] = \frac{1}{2}. \qquad \text{(+2 points)}$$

For the average power

$$\begin{aligned}\int_{T_0} |x(t)|^2\, dt &= \int_0^{1/2} 4t^2\, dt + \int_{1/2}^{1} (2-2t)^2\, dt && \text{(+1 point)}\\ &= 2\int_0^{1/2} 4t^2\, dt \\ &= \frac{8}{3}\, t^3\Big|_{t=0}^{1/2} \\ &= \frac{8}{3}\frac{1}{8} = \frac{1}{3} && \text{(+1 point)}\end{aligned}$$

(d) (2 points (bonus)) Show that

$$\sum_{n=-\infty}^{\infty} \frac{4}{(2n+1)^4\pi^4} = \frac{1}{12}.$$

Solution: Note that

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{4}{(2n+1)^4\pi^4} &= \sum_{n=-\infty}^{\infty} \hat{X}[n]^2 - \hat{X}[0]^2 && \text{(+1 bonus point)}\\ &= \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt - \hat{X}[0]^2 \\ &= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}. && \text{(+1 bonus point)}\end{aligned}$$

2. **Continuous-time System**

A causal continuous-time linear time-invariant system is described by the ODE

$$\dot{y} + y = \dot{u}.$$

(a) (2 points) Calculate the system's transfer-function, $G(s)$, and its region of convergence.

Solution: By inspection

$$G(s) = \frac{s}{s+1} \qquad \text{(+1 point)}$$

The region of convergence is $\mathrm{Re}\{s\} > -1$. (+1 point)

(b) (1 point) Is the system asymptotically stable? Explain.

Solution: Yes because the pole of $G(s)$ is $s = -1$ which has negative real part. (+1 point)

ALTERNATIVELY, yes because its region of convergence contains the imaginary axis.

(c) (2 points) Calculate its impulse response, $g(t)$. *Hint: invert a transform!*

Solution:

Using the Fourier transform:

$$\begin{aligned} g(t) &= \mathcal{F}^{-1}\{G(j\omega)\} \\ &= \mathcal{F}^{-1}\left\{1 - \frac{1}{j\omega + 1}\right\} \\ &= \delta(t) - e^{-t}1(t). \end{aligned} \qquad \text{(+2 points)}$$

ALTERNATIVELY, using the Laplace transform:

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\{G(s)\} \\ &= \mathcal{L}^{-1}\left\{1 - \frac{1}{s + 1}\right\} \\ &= \delta(t) - e^{-t}1(t). \end{aligned} \qquad \text{(+2 points)}$$

(d) (1 point) Without performing any calculations, explain whether or not $G(j\omega) = \mathcal{F}\{g(t)\}$?

Solution: $G(j\omega) = \mathcal{F}\{g(t)\}$ because the region of convergence of $G(s)$ includes the imaginary axis $s = j\omega$. (+1 point)

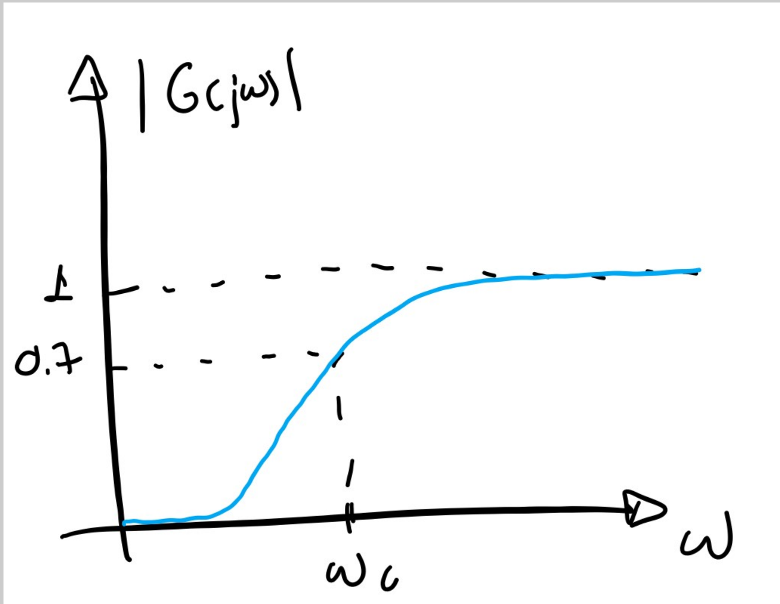
ALTERNATIVELY, because $G(s)$ is asymptotically stable.

(e) (4 points) Calculate the system's frequency response $G(j\omega)$ and sketch its magnitude and phase. What kind of filter response does this system display?

Solution: The frequency response is

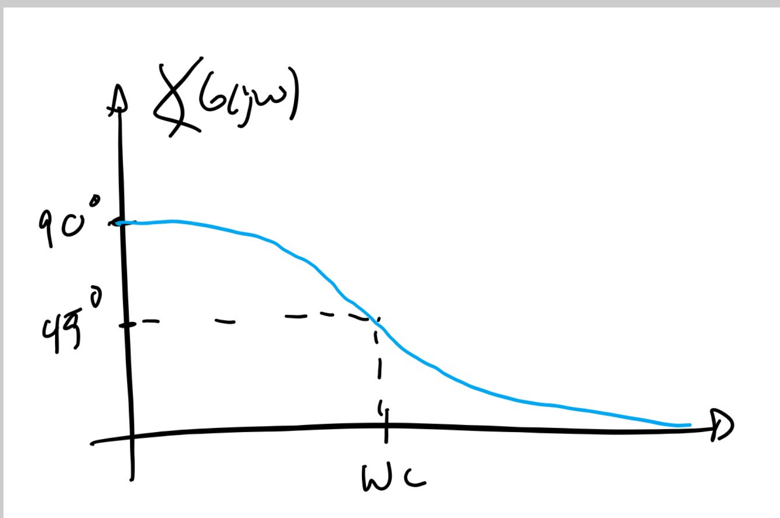
$$G(j\omega) = \frac{j\omega}{j\omega + 1} = \frac{|\omega|}{\sqrt{\omega^2 + 1}} e^{j(\pi/2 - \tan^{-1}\omega)} \qquad \text{(+1 point)}$$

The magnitude should look like:



(+1 point)

The phase should look like:



(+1 point)

The system is a high-pass filter. (+1 point)

3. **System Response**

   (a) (4 points) Use either the Fourier or the Laplace transform to calculate the output response, $y(t)$, of the system from Question 2 to the input $u(t) = \mathrm{rect}(t - 1/2)$. Sketch the response.

Solution: Using the Fourier transform. First calculate the transform of the input

$$U(j\omega) = \mathcal{F}\{\mathrm{rect}(t - 1/2)\} = e^{-j\omega/2}\,\mathrm{sinc}(\omega/(2\pi)) \qquad \text{(+1 point)}$$

(see the Laplace solution for an alternative formula) and the transform of the output

$$Y(j\omega) = G(j\omega)U(j\omega) = \frac{j\omega}{j\omega + 1} e^{-j\omega/2}\,\mathrm{sinc}(\omega/(2\pi)) \qquad \text{(+1 point)}$$

from which

$$
\begin{aligned}
y(t) &= \mathcal{F}^{-1}\{Y(j\omega)\} \\
&= \mathcal{F}^{-1}\left\{\frac{j\omega}{j\omega+1}e^{-j\omega/2}\frac{e^{j\omega/2}-e^{-j\omega/2}}{2j\omega/2}\right\} \\
&= \mathcal{F}^{-1}\left\{\frac{j\omega}{j\omega+1}\frac{1-e^{-j\omega}}{j\omega}\right\} \\
&= \mathcal{F}^{-1}\left\{\frac{1-e^{-j\omega}}{j\omega+1}\right\} \\
&= \mathcal{F}^{-1}\left\{\frac{1}{j\omega+1}\right\} - \mathcal{F}^{-1}\left\{e^{-j\omega}\frac{1}{j\omega+1}\right\} \\
&= e^{-t}1(t) - e^{-(t-1)}1(t-1).
\end{aligned}
$$

(+1 point)

ALTERNATIVELY: Using the Laplace transform. First calculate the transform of the input

$$U(s) = \mathcal{L}\{\mathrm{rect}(t-1/2)\} = \mathcal{L}\{1(t) - 1(t-1)\} = \frac{1-e^{-st}}{s}$$ (+1 point)

and the transform of the output

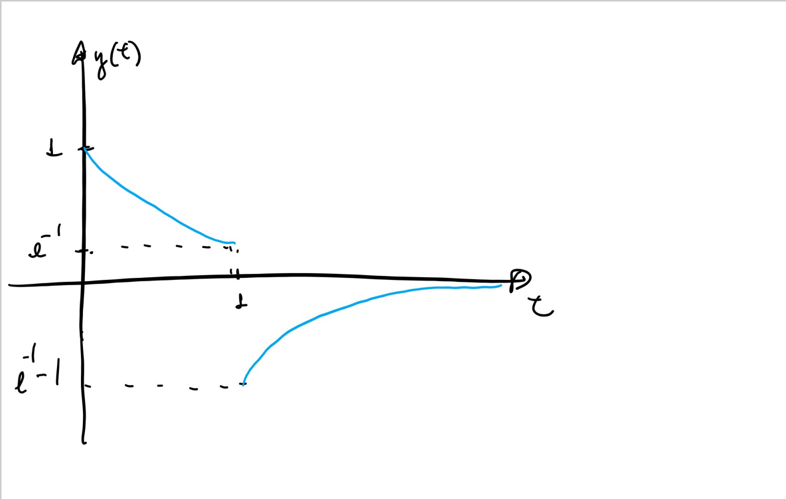
$$Y(s) = G(s)U(s) = \frac{s}{s+1}\frac{1-e^{-s}}{s} = \frac{1-e^{-s}}{s+1}$$ (+1 point)

from which

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{1-e^{-s}}{s+1}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{F}^{-1}\left\{e^{-s}\frac{1}{s+1}\right\} \\
&= e^{-t}1(t) - e^{-(t-1)}1(t-1).
\end{aligned}
$$

(+1 point)

A sketch of the solution should look like:



(+1 point)

(b) (2 points) Calculate the coefficients of the Fourier series of the response of the system from Question 2 to the input from Question 1 with period $T_0 = 2\pi/10$.

Solution: Changing the period does not change the coefficients of the Fourier series of a signal. Therefore

$$\hat{Y}[n] = G(j\omega_0 n)\hat{X}[n] = \frac{j\omega_0 n}{j\omega_0 n + 1}\hat{X}[n], \quad \omega_0 = \frac{2\pi}{2\pi/10} = 10 \qquad \text{(+2 points)}$$

in which $\hat{X}[n]$ is given in Question 1.

(c) (2 points) Calculate and sketch the magnitude of the first 4 nonzero coefficients of the Fourier series of the input and the magnitude of the first 4 nonzero coefficients of the Fourier series of the output.

Solution:

$$|\hat{X}[n]| = \begin{cases} \frac{1}{2}, & n = 0, \\ 0, & n \text{ even}, \\ \frac{2}{n^2\pi^2}, & n \text{ odd} \neq 0 \end{cases}$$

The first 4 nonzero coefficients are

$$|\hat{X}[0]| = \frac{1}{2}, \quad |\hat{X}[1]| = \frac{2}{\pi^2} \approx 0.2, \quad |\hat{X}[3]| = \frac{2}{9\pi^2} \approx 0.02, \quad |\hat{X}[5]| = \frac{2}{25\pi^2} \approx 0.008,$$

(+1/2 point)

From part (b)

$$|\hat{Y}[n]| = \frac{|10n|}{\sqrt{(10n)^2 + 1}}|\hat{X}[n]|$$

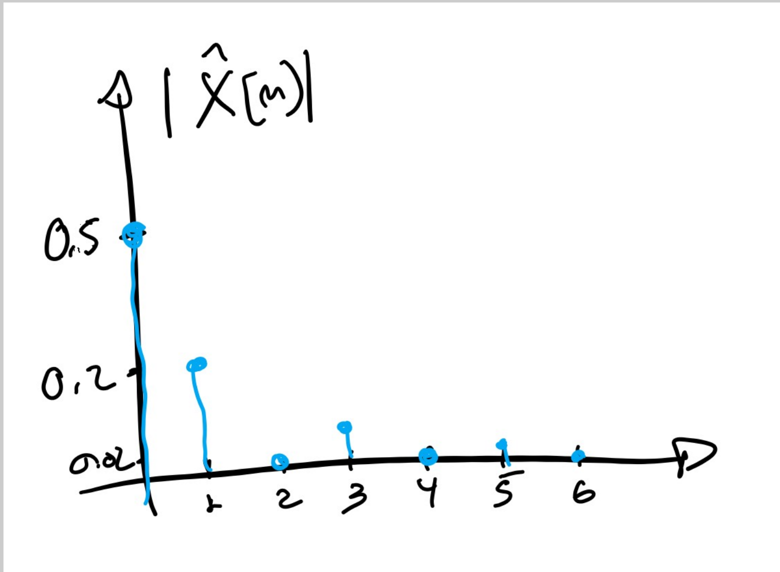
from which

$$|\hat{Y}[0]| = 0, \qquad |\hat{Y}[1]| \approx 0.2, \qquad |\hat{Y}[3]| \approx 0.02, \qquad |\hat{Y}[5]| \approx 0.008, \qquad \text{(+1/2 point)}$$

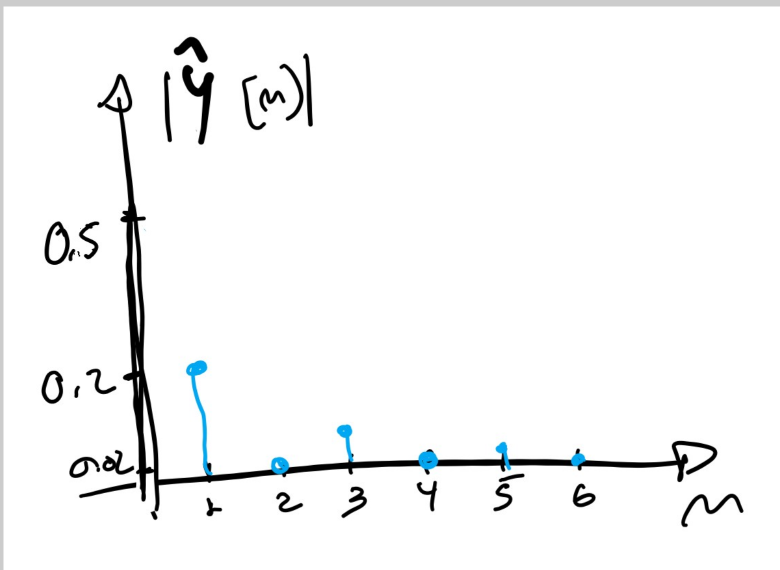
because

$$\frac{|10n|}{\sqrt{(10n)^2 + 1}} \geq \frac{10}{\sqrt{10^2 + 1}} \approx 0.99, \quad n \geq 1.$$

The sketches of the magnitude of the coefficients should look like:



(+1/2 point)



(+1/2 point)

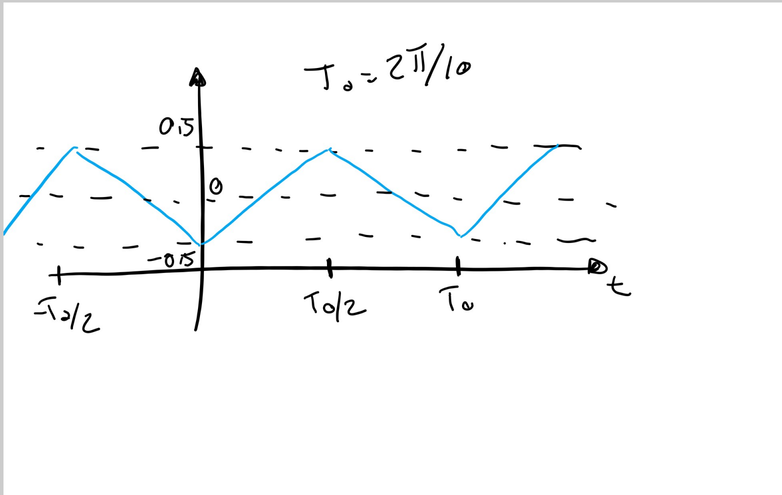
(d) (2 points) Based on your answers so far, are you able to approximately sketch the output signal? *Hint: compare the coefficients of the input with the coefficients of the output.*

Solution: The only coefficient magnitude that gets modified by the filter is the $n = 0$ coefficient. A quick look at the phase

$$\angle G(j10n) = 90^\circ - \tan^{-1} 10n \leq 90^\circ - 84.3 \approx 5^\circ, \quad n \geq 1$$

suggests that the only difference between the input and output signal will be due to the change in the $n = 0$ coefficient. That is the output signal should have approximately the same shape but zero average. (+1 point)

A sketch of the output should therefore be approximately



(+1 point)

4. **Filter Transformations**

Answer the following questions.

(a) (4 points (bonus)) Let $G(s)$ be a proper asymptotically stable rational transfer-function of order n for which $|G(0)| > 0$ and $|G(j)|^2 = 1/2$. Show that

$$\tilde{G}(s) = G(\omega_c/s), \quad \omega_c > 0$$

is such that $\tilde{G}(s)$ is a proper asymptotically stable rational transfer-function of order n and that

$$|\tilde{G}(0)| = \lim_{\omega\to\infty} |G(j\omega)|^2, \qquad |\tilde{G}(j\omega_c)|^2 = |G(j)|^2 = 1/2, \qquad \lim_{\omega\to\infty} |\tilde{G}(j\omega)| = |G(0)|.$$

Furthermore, if the degree of the denominator of $G(s)$ is n and the degree of the numerator is $m \le n$ then $\tilde{G}(s)$ has $n-m$ zeros at zero.

Hint: use a generic function of the form $G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$.

Solution: Consider a generic $G(s)$ of the form

$$G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}, \quad m \le n.$$

Therefore

$$\begin{aligned}\tilde{G}(s) = G(\omega_c/s) &= \frac{b_0(\omega_c/s)^m + b_1(\omega_c/s)^{m-1} + \cdots + b_m}{(\omega_c/s)^n + a_1(\omega_c/s)^{n-1} + \cdots + a_n} \\ &= \frac{s^n b_m + \cdots + s^{n-m+1}\omega_c^{m-1} b_1 + s^{n-m}\omega_c^m b_0}{s^n a_n + \cdots + s\omega_c^{n-1} a_1 + \omega_c^n}\end{aligned}$$

is a proper rational function of order n. Let $s = p_0$ be a pole of $G(s)$. (+1 point)

This also shows that $\tilde{G}(s)$ has $n-m$ zeros at zero because

$$\tilde{G}(s) = \frac{s^{n-m}\left(s^m b_m + \cdots + s\omega_c^{m-1} b_1 + \omega_c^m b_0\right)}{s^n a_n + \cdots + s\omega_c^{n-1} a_1 + \omega_c^n}.$$ (+1 point)

If $G(s)$ is asymptotically stable then $p_0 \neq 0$ and $s = \omega_c/p_0$ is a pole of $\tilde{G}(s)$. Note that if $p_0 = \alpha + j\beta$ then

$$\mathrm{Re}\{\omega_c/p_0\} = \mathrm{Re}\left\{\frac{\omega_c \alpha}{\alpha^2+\beta^2} - j\frac{\omega_c \beta}{\alpha^2+\beta^2}\right\} = \frac{\omega_c \alpha}{\alpha^2+\beta^2}$$

which is negative if $\alpha < 0$. Hence $\tilde{G}(s)$ is also asymptotically stable. (+1 point)

The given relationships for the gains are true because

$$\lim_{\omega\to\bar{\omega}} |\tilde{G}(j\omega)| = \lim_{\omega\to\bar{\omega}} |G(-j\omega_c/\omega)| = \lim_{\omega\to\bar{\omega}} |G(j\omega_c/\omega)|. \qquad \text{(+1 point)}$$

(b) (1 point) Let

$$G(s) = \frac{1}{s+1}, \quad \mathrm{Re}\{s\} > -1$$

and calculate

$$|G(j0)|, \qquad |G(j1)|^2, \qquad \lim_{\omega\to\infty} |G(j\omega)|.$$

Solution: Simply evaluate

$$|G(j0)| = 1, \quad |G(j1)|^2 = \frac{1}{|j+1|^2} = \frac{1}{2}, \quad \lim_{\omega\to\infty} |G(j\omega)| = \lim_{\omega|\to\infty} \frac{1}{\sqrt{\omega^2+1}} = 0 \quad \text{(+1 points)}$$

(c) (2 points) Apply the transformation from part (a) to the transfer-function in part (b) to calculate $\tilde{G}(s)$ and

$$|\tilde{G}(j0)|, \qquad |\tilde{G}(j\omega_c)|^2, \qquad \lim_{\omega\to\infty} |G(j\omega)|.$$

Solution:

$$\tilde{G}(s) = \frac{1}{\omega_c/s+1} = \frac{s}{s+\omega_c} \qquad \text{(+1 point)}$$

and

$$|\tilde{G}(j0)| = \lim_{\omega\to\infty} |G(j\omega)| = 0, \quad |\tilde{G}(j\omega_c)|^2 = |G(j1)|^2 = \frac{1}{2}, \quad \lim_{\omega\to\infty} |G(j\omega)| = |G(0)| = 1$$

(+1 points)