

MAE143 B - Linear Control - Spring 2012
Final, June 11th

Instructions:

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities
2. You have 170 minutes
3. Many questions show the answer which you're asked to prove. Use the given answers to move on to the next question. Do not get stuck!
4. Write your answers on your "Blue Book"
5. Do not forget to write your name, student number, and instructor
6. This exam has 8 questions, for a total of 58 points and 14 bonus points

Questions:

1. Root Locus

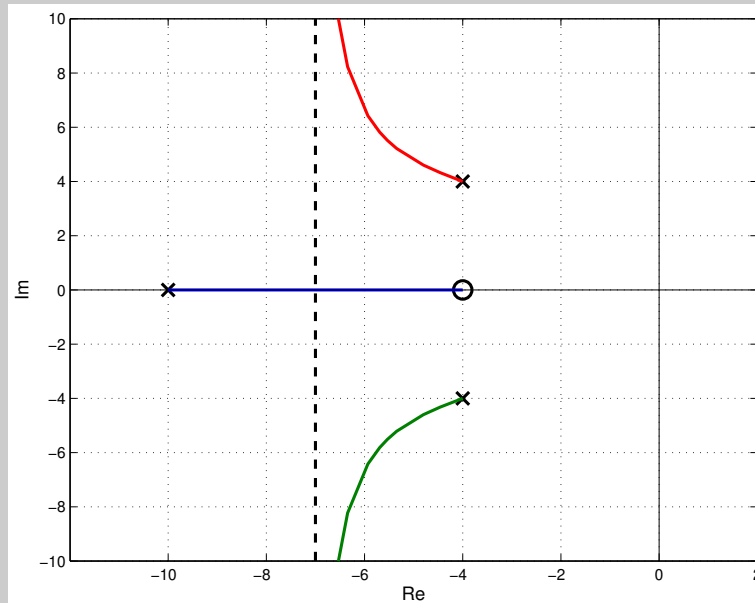
Answer the questions based on the diagrams in the attached Figure 2, which shows the open-loop poles and zeros of a linear system marked with 'x's (poles) and 'o's (zeros). Make sure you return the diagrams with your blue book.

Hint: you only need the plots to answer the questions, no other computations are necessary!

- (a) [4 points] Use the attached graph sheet Figure 2(a) to sketch the root-locus associated with the open-loop zeros and poles shown in the figure. Is there a value of $K > 0$ for which closing a negative unit feedback loop with a proportional gain K leads to an unstable closed-loop system?

Solution:

The root locus should look like:



with two asymptotes starting at 90° and intersecting at

$$\frac{-10 - 4 - 4 + 4}{3 - 1} = -\frac{14}{2} = -7$$

Because the root-locus never crosses the imaginary axis there are no $K > 0$ for which the closed-loop system is unstable. (+1.0 point)

NOTE TO GRADERS:

Correct real root locus (+1.0 point)

Correct number and location of asymptotes (no need to be exact) (+1.0 point)

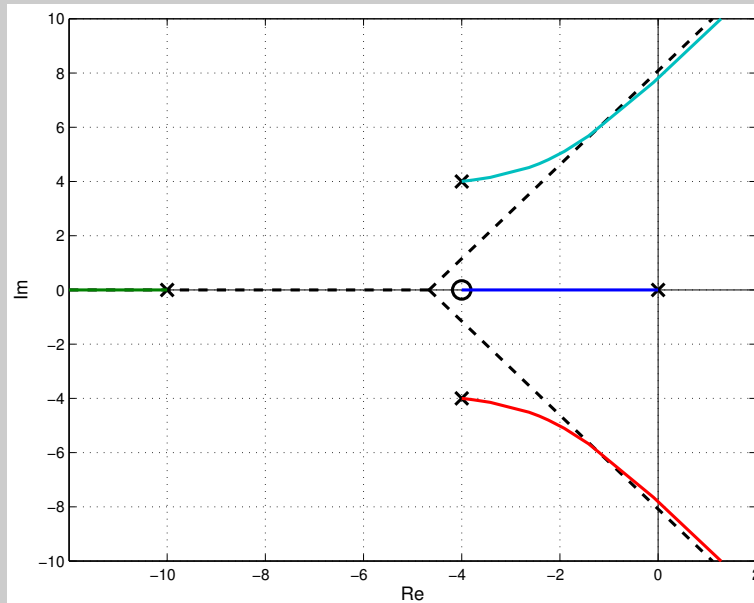
Correct look of complex branches (+1.0 point)

- (b) [4 points] Now consider the same system in closed-loop with an integral controller $K(s) = Ks^{-1}$. Use the attached graph sheet Figure 2(b) to sketch the associated root-locus. Is there a value of $K > 0$ for which the closed-loop system is unstable?

Solution:

The integrator adds a pole at $s = 0$. (+1.0 point)

The root locus should look like:



with three asymptotes starting at 60° and intersecting at

$$\frac{-10 - 4 - 4 - 0 + 4}{4 - 1} = -\frac{14}{3} \approx -4.7$$

Because the root-locus will cross the imaginary axis near the two asymptotes there is $K > 0$ for which the closed-loop system is unstable. (+1.0 point)

NOTE TO GRADERS:

Correct real root locus (+1.0 point)

Correct number and location of asymptotes (no need to be exact) (+1.0 point)

- (c) [4 points] Consider the original system in closed-loop with an integral plus proportional controller $K(s) = K_p + K_i s^{-1}$ in which $K_i = 2K_p$. Use the attached graph sheet Figure 2(c) to sketch the associated root-locus. Is there a value of $K_p > 0$ for which the closed-loop system is unstable?

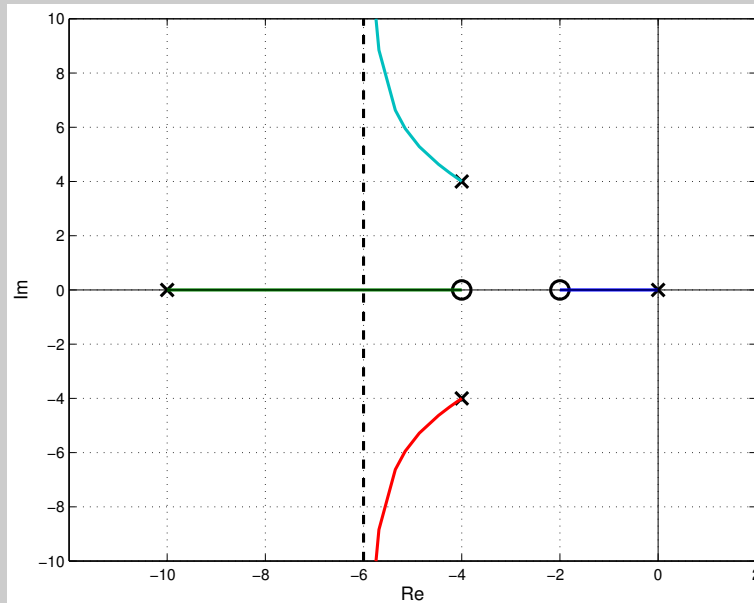
Solution:

The controller is

$$K(s) = K_p + \frac{K_i}{s} = K_p \frac{s + K_i/K_p}{s} = K_p \frac{s + 2}{s}$$

which adds a pole at $s = 0$ and a zero at $s = -2$. (+1.0 point)

The root locus should look like:



with three asymptotes starting at 60° and intersecting at

$$\frac{-10 - 4 - 4 - 0 + 4 + 2}{4 - 2} = -\frac{12}{2} = -6$$

Because the root-locus will never cross the imaginary axis there is no $K_p > 0$ for which the closed-loop system is unstable. (+1.0 point)

NOTE TO GRADERS:

Correct real root locus (+1.0 point)

Correct number and location of asymptotes (no need to be exact) (+1.0 point)

- (d) [4 points] Finally consider the original system in closed-loop with a proportional plus derivative controller $K(s) = K_p + K_d s$ in which $K_p = 2K_d$. Use the attached graph sheet Figure 2(d) to sketch the associated root-locus. Is there a value of $K_d > 0$ for which the closed-loop system is unstable?

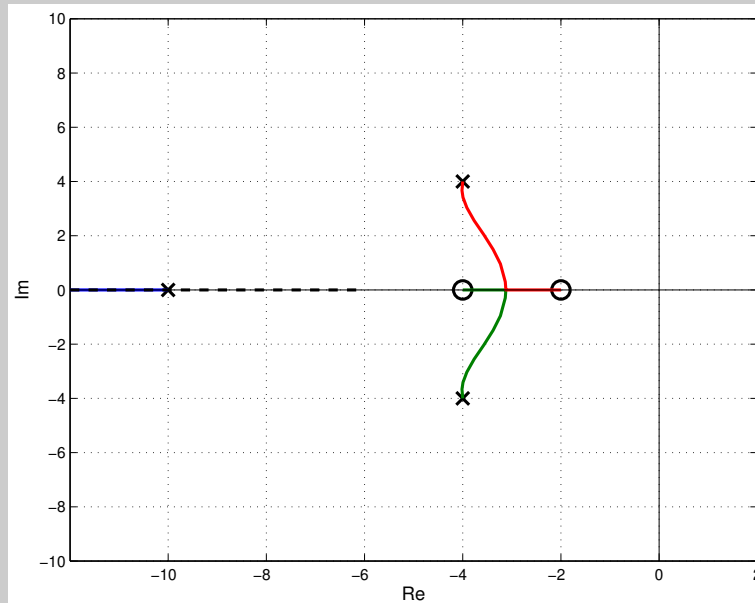
Solution:

The controller is

$$K(s) = K_p + K_d s = K_d(s + K_p/K_d) = K_d(s + 2)$$

which adds a zero at $s = -2$. (+1.0 point)

The root locus should look like:



with three asymptotes starting at 60° and intersecting at

$$\frac{-10 - 4 - 4 + 2 + 4}{4 - 2} = -\frac{12}{2} \approx -6$$

Because the root-locus will never cross the imaginary axis there is no $K_d > 0$ for which the closed-loop system is unstable. (+1.0 point)

NOTE TO GRADERS:

Correct real root locus (+1.0 point)

Correct number and location of asymptotes (no need to be exact) (+1.0 point)

Total regular points: [16]

2. Bode Diagrams

Answer the next questions for each of the transfer-functions:

$$F_1(s) = \frac{\alpha}{s + \alpha},$$

$$H_1(s) = \frac{0.01}{(s + \alpha)(s + 0.2)},$$

$$F_2(s) = \frac{\alpha\beta s + \alpha^2}{s^2 + \alpha\beta s + \alpha^2},$$

$$H_2(s) = \frac{0.01s}{(s^2 + \alpha\beta s + \alpha^2)(s + 0.2)}.$$

- (a) [4 points] Enumerate the poles and zeros and calculate ω_n and ζ for each complex pole or zero;

Solution:

$F_1(s)$: pole = $-\alpha$, no zeros; (+1.0 point)

$F_2(s)$: poles = $-\omega_n\zeta \pm \omega_n\sqrt{\zeta^2 - 1}$, zero = $-\alpha/\beta$, $\omega_n = \alpha$, $\zeta = \beta/2$

$$\begin{aligned}
H_1(s): \text{poles} &= -\alpha, -0.2, \quad \text{no zeros;} & (+1.0 \text{ point}) \\
H_2(s): \text{poles} &= -0.2, -\omega_n \zeta \pm \omega_n \sqrt{\zeta^2 - 1}, \quad \text{zero} = 0, \quad \omega_n = \alpha, \quad \zeta = \beta/2 & (+1.0 \text{ point}) \\
& & (+1.0 \text{ point})
\end{aligned}$$

- (b) [4 points] Rewrite the transfer-function in a form suitable for Bode plots in which each pole and zero has unit gain;

Solution:

Each pole or zero should have unit gain:

$$F_1(s) = \frac{1}{\frac{s}{\alpha} + 1} \quad (+1.0 \text{ point})$$

$$F_2(s) = \frac{\frac{\beta s}{\alpha} + 1}{\left(\frac{s}{\alpha}\right)^2 + 2\frac{\beta}{2}\frac{s}{\alpha} + 1}, \quad (+1.0 \text{ point})$$

$$H_1(s) = \frac{0.05}{\alpha} \frac{1}{\left(\frac{s}{\alpha} + 1\right) \left(\frac{s}{0.2} + 1\right)}, \quad (+1.0 \text{ point})$$

$$H_2(s) = \frac{0.05}{\alpha^2} \frac{s}{\left(\left(\frac{s}{\alpha}\right)^2 + 2\frac{\beta}{2}\frac{s}{\alpha} + 1\right) \left(\frac{s}{0.2} + 1\right)} \quad (+1.0 \text{ point})$$

- (c) [4 points] Evaluate the magnitude and phase at $s = j0$, and $s \rightarrow j\infty$.

Solution:

As they are all strictly proper all gains at ∞ are zero

$$|F_1(0)| = 1 \quad |F_1(j\infty)| = 0 \quad (+0.5 \text{ point})$$

$$|F_2(0)| = 1, \quad |F_2(j\infty)| = 0 \quad (+0.5 \text{ point})$$

$$|H_1(0)| = \frac{0.05}{\alpha}, \quad |H_1(j\infty)| = 0 \quad (+0.5 \text{ point})$$

$$|H_2(0)| = \frac{0.05}{\alpha^2}, \quad |H_2(j\infty)| = 0 \quad (+0.5 \text{ point})$$

Phase at 0 is determined by the number of poles or zeros at 0 and phase at ∞ is determined by the difference between the number of poles and number of zeros.

$$\angle F_1(0) = 0 \quad \angle F_1(j\infty) = -90^\circ \quad (+0.5 \text{ point})$$

$$\angle F_2(0) = 0, \quad \angle F_2(j\infty) = -90^\circ \quad (+0.5 \text{ point})$$

$$\angle H_1(0) = 0, \quad \angle H_1(j\infty) = -180^\circ \quad (+0.5 \text{ point})$$

$$\angle H_2(0) = +90^\circ, \quad \angle H_2(j\infty) = -180^\circ \quad (+0.5 \text{ point})$$

(points shown are total for all four transfer-functions)

Then answer this question:

- (d) [4 points] Identify each Bode plot in Figure 3 with one of the transfer-functions. Provide evidence that all plots were drawn with $\alpha = 1$ and $\beta = 0.5$.
Hint: take advantage of the asymptotes shown in the plots

Solution:

- (a) $H_2(s)$ is the only function with $\angle H_2(0) = +90^\circ$; (+0.5 point)
 (b) $F_1(s)$ is the only function with a single pole; (+0.5 point)
 (c) $F_2(s)$ is the other function with $|F_2(0) = 1|$ (+0.5 point)
 (d) $H_1(s)$ by exclusion. (+0.5 point)

Based on the inflexion points of the asymptotes:

- (a) $H_1(s)$ has a complex pole at $s = \omega_n = \alpha = 1$;
 hump suggests β small; (+0.5 point)
 (b) $F_1(s)$ has a single pole at $s = -\alpha = -1$; (+0.5 point)
 (c) $F_2(s)$ has a complex pole at $s = \omega_n = \alpha = 1$
 and a zero at $s = \alpha/\beta = \beta^{-1} = 2$ (+0.5 point)
 (d) $H_1(s)$ has a pole at $s = -\alpha = -1$; (+0.5 point)

Total regular points: [16]

3. Free Falling Body

You will now derive a simple model for a free falling body using Newton's law.

- (a) [2 points] The dynamics of a free falling body is given by the linear ODE

$$m\ddot{x} + b\dot{x} = mg$$

where x is the vertical distance measured from the start of the fall, g is the gravitational acceleration, $m > 0$ is the body mass, $b \geq 0$ is the damping coefficient. Assuming zero initial conditions, show that

$$X(s) = G(s) R(s), \quad G(s) = \frac{1}{s(ms + b)},$$

where $X(s)$ is the Laplace transforms of the signal $x(t)$, and

$$R(s) = \frac{mg}{s}.$$

For the next questions consider $m = 100$ Kg, $b = 20$ Kg/s, and $g = 10$ m/s².

Solution:

Applying Laplace transforms with zero initial conditions

$$ms^2X(s) + bsX(s) = \frac{mg}{s} \quad (+1.0 \text{ point})$$

from where

$$X(s) = \frac{1}{s(ms + b)} \frac{mg}{s} = G(s)R(s). \quad (+1.0 \text{ point})$$

- (b) [5 points] Assuming zero initial conditions, determine $x(t)$.

Solution:

With zero initial conditions

$$X(s) = G(s)R(s) = \frac{mg}{s^2(ms + b)} \quad (+1.0 \text{ point})$$

Expanding in partial fractions

$$X(s) = \frac{k_1}{s^2} + \frac{k_2}{s} + \frac{k_3}{s + b/m}$$

where

$$k_1 = \lim_{s \rightarrow 0} s^2 X(s) = \frac{mg}{b} = 50$$

$$k_2 = \lim_{s \rightarrow 0} (d/ds) s^2 X(s) = \lim_{s \rightarrow 0} -\frac{g}{(s + b/m)^2} = -\frac{m^2 g}{b^2} = -250$$

$$k_3 = \lim_{s \rightarrow 0} (s + b/m) X(s) = \frac{m^2 g}{b^2} = 250 \quad (+3.0 \text{ points})$$

Applying Laplace inverse

$$x(t) = \mathcal{L}^{-1} X(s) = [50t + 250(1 - e^{-.2t})] 1(t) \quad (+1.0 \text{ point})$$

- (c) [2 points] Compute the maximum velocity attained by the body.

Solution:

Many possibilities.

From part (b):

$$\dot{x}(t) = [50 - 50e^{-.2t}] 1(t) \quad (+1.0 \text{ point})$$

and

$$\lim_{t \rightarrow \infty} \dot{x}(t) = 50 \text{ m/s} \quad (+1.0 \text{ point})$$

Alternatively.

From the differential equation with a constant velocity $\dot{x} = v$ we have $\ddot{x} = 0$ and

$$bv = mg \quad (+1.0 \text{ point})$$

or

$$v = \frac{mg}{b} = 50 \text{ m/s} \quad (+1.0 \text{ point})$$

Total regular points: [9]

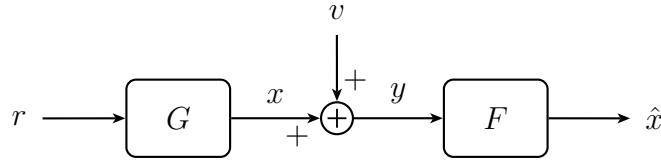


Figure 1: Block-diagram for Question 4

You completed 41 out of 58 possible points. On the next questions you will have to do some reading to solve an interesting problem and earn 17 points. Each question is designed to guide you in the process. It is about a problem known as *estimation*.

Estimators are filters used to recover signals that might have been transformed by systems or corrupt by other signals. A typical scenario is the one in Figure 1. The signal of interest is x , which is produced by the system G as a result of another signal r .

Take for instance the example of the free falling body: mass and gravity produce a constant force (the signal $r(t) = mg, t \geq 0$) which generates a trajectory (the signal $x(t)$) which we want to estimate. The estimate (the signal $\hat{x}(t)$) should be constructed based on a measurement $y(t)$, which contains information about the trajectory $x(t)$ although corrupted by the disturbance signal $v(t)$.

We will assume that r is constant but unknown and is not been directly measured, for example, the mass of the body might not be known (when r is know or is been measured, we can build better estimators that use r and y as inputs and do not look like Figure 1). Back to Figure 1, the simplest possible estimator is $F = 1$, in which case $\hat{x} = y$. In the case of the free falling body, r is constant (low frequency), and the disturbance v is unknown but high frequency (e.g. a sinusoid of amplitude $\bar{v}$ and frequency λ). It is natural to expect that some form of low-pass filter will produce a better estimate than $F = 1$. It is in this context that the next questions have been formulated.

4. Estimator Analysis

Answer the following questions regarding the signals and systems interconnected as in the block-diagram shown in Figure 1:

- (a) [2 points] Show that the estimate $\hat{x}$ is related to the signals r and v by

$$\hat{x} = Fv + FGr,$$

and that the estimation error, e , is

$$e = x - \hat{x} = Hr - Fv,$$

where $H = (1 - F)G$.

Solution:

From the diagram

$$\begin{aligned}\hat{x} &= Fy \\ &= F(v + x) \\ &= Fv + FGr\end{aligned}\quad (+1.0 \text{ point})$$

and

$$\begin{aligned}e &= x - \hat{x} \\ &= Gr - Fv - FGr \\ &= (1 - F)Gr - Fv\end{aligned}\quad (+1.0 \text{ point})$$

A good estimator will have small estimation error e . We will compute a measure of the size of e in the next questions. From this point on, assume that r is constant and the measurement disturbance v is a sinusoid, i.e.

$$r(t) = \bar{r}, \quad v(t) = \bar{v} \cos(\lambda t), \quad t \geq 0, \quad (\text{RV})$$

of amplitude $\bar{r}$ and $\bar{v}$ and frequency $\lambda > 0$.

- (b) [3 points] Assume that both F and H are stable rational transfer functions, that is F and H have poles with negative real part. Show that

$$e_{\text{ss}}(t) = \bar{r} H(0) - \bar{v} |F(j\lambda)| \cos(\lambda t + \angle F(j\lambda))$$

is the steady-state response of the estimation error to the inputs (RV).

Hint: Articulate your arguments based on linearity and frequency response.

Solution:

Because $H(s)$ is stable we can use the frequency response to compute the steady state response to a constant inputs $r(t) = \bar{r}$ as

$$e_{\text{ss}}^r(t) = \bar{r} H(0) \quad (+1.0 \text{ point})$$

when $v = 0$. On the other hand, when $r = 0$, because $f(s)$ is stable, the steady-state response to $v(t) = \bar{v} \cos \lambda t$ will be

$$e_{\text{ss}}^v(t) = -\bar{v} |F(j\lambda)| \cos(\lambda t + \angle F(j\lambda)). \quad (+1.0 \text{ point})$$

By linearity

$$e_{\text{ss}}(t) = e_{\text{ss}}^r(t) + e_{\text{ss}}^v(t) = \bar{r} H(0) - \bar{v} |F(j\lambda)| \cos(\lambda t + \angle F(j\lambda)). \quad (+1.0 \text{ point})$$

- (c) [1 point (bonus)] Show that $|e_{\text{ss}}(t)| \leq |H(0)||\bar{r}| + |F(j\lambda)||\bar{v}|$.

Solution:

Start by taking absolute values

$$\begin{aligned}
 |e_{ss}(t)| &= |\bar{r} H(0) - \bar{v} |F(j\lambda)| \cos(\lambda t + \angle F(j\lambda))| \\
 &\leq |\bar{r}| |H(0)| + |\bar{v}| |F(j\lambda)| |\cos(\lambda t + \angle F(j\lambda))| \\
 &\leq |\bar{r}| |H(0)| + |\bar{v}| |F(j\lambda)| \quad (+1.0 \text{ point})
 \end{aligned}$$

Total regular points: [5]
 Bonus points: [1]

You will now use Questions 3 and 4 to design estimators for the vertical position of a free falling body. Refer to the block-diagram in Figure 1 and the transfer-function G derived in Question 3. We will start with the simplest possible estimator, $F = 1$. Assume that the inputs are the constant and sinusoid in (RV) with $\bar{r} = mg$.

5. Estimator Design I

(a) [2 points] Let the estimator F be simply

$$F = F_0 = 1, \quad \text{with which} \quad H = H_0 = [1 - F_0] G = 0.$$

Use Part (c) in Question 4 to show that the input (RV) produces a steady-state estimation error

$$|x_{ss}(t) - \hat{x}_{ss}(t)| \leq |\bar{v}|.$$

Explain the result.

Solution:

From Question 4 Part (c)

$$|x_{ss}(t) - \hat{x}_{ss}(t)| = |e_{ss}(t)| \leq |H(0)| |\bar{r}| + |F(j\lambda)| |\bar{v}| = |\bar{v}|$$

because $H(0) = H = 0$ and $F(j\lambda) = F = 1$. (+1.0 point)

As the filter is simply $\hat{x} = y = x + v$, we have $\hat{x} - x = v = \bar{v} \cos \lambda t$ and hence $|e| = |\hat{x} - x| \leq |\bar{v}|$. (+1.0 point)

Total regular points: [2]

Your goal is to reduce the amplitude of the estimation error, $|x - \hat{x}|$, below the amplitude of the perturbation $\bar{v}$. Because r is low frequency (constant) and v is high-frequency, the next estimator will be a first-order low-pass filter.

6. Estimator Design II

(a) [2 points] Let the estimator F be the first-order low pass filter

$$F(s) = F_1(s) = \frac{\alpha}{s + \alpha}, \quad \alpha > 0.$$

Show that

$$H(s) = H_1(s) = [1 - F_1(s)] G(s) = \frac{0.01}{(s + \alpha)(s + 0.2)}.$$

Solution:

Compute

$$\begin{aligned} H(s) &= [1 - F(s)]G(s) \\ &= \left[1 - \frac{\alpha}{s + \alpha}\right] \frac{1}{s(ms + b)} \\ &= \frac{s}{s + \alpha} \frac{1}{s(ms + b)} \\ &= \frac{1/m}{(s + \alpha)(s + b/m)}. \end{aligned} \quad (+2.0 \text{ point})$$

(b) [2 points (bonus)] Use Part (c) in Question 4 to show that

$$|x_{ss}(t) - \hat{x}_{ss}(t)| \leq \frac{50}{\alpha} + \frac{\alpha|\bar{v}|}{\sqrt{\lambda^2 + \alpha^2}}.$$

What does the first term mean? Why does it get bigger as α gets smaller?

Hint: Look at the solution $x(t)$ in Question 3 Part (b)

Solution:

From Question 4 Part (c)

$$|e_{ss}(t)| \leq |H(0)||\bar{r}| + |F(j\lambda)||\bar{v}|$$

We compute $\bar{r} = mg = 1000$,

$$H(0) = \frac{1/m}{(s + \alpha)(s + b/m)} = \frac{1/100}{20/100\alpha} = \frac{1}{20} \frac{1}{\alpha} = \frac{0.05}{\alpha}$$

and

$$|F(j\lambda)| = \frac{\alpha}{|j\lambda + \alpha|} = \frac{\alpha}{\sqrt{\lambda^2 + \alpha^2}} \quad (+1.0 \text{ point})$$

The first term comes from not been able to track $x(t)$ produced by a constant $\bar{r}$ without error steady-state error. $x(t)$ has a component that grows with t and the filter introduces a *bias*. The smaller the α the larger the bias because the response of the filter, which has a pole at $s = -\alpha$, gets “slower”. (+1.0 point)

(c) [3 points] Use the formula in Part (b) to compute a bound for $|x_{ss}(t) - \hat{x}_{ss}(t)|$ when $\bar{v} = 5 \text{ m}$, and $\lambda = 10 \text{ rad/s}$, and α takes the values

$$\alpha = 1, \quad \alpha = 10, \quad \alpha = 100.$$

What values of α lead to the smallest estimation error? Is there any value of α for which the estimation error is smaller than $\bar{v} = 5$ m? Is that what you expected?

Solution:

From Part (b)

$$|e_{ss}(t)| \leq \frac{50}{\alpha} + \frac{\alpha \bar{v}}{\sqrt{\lambda + \alpha^2}}$$

When $\bar{v} = 5$, $\lambda = 10$ and $\alpha = 100$

$$|e_{ss}(t)| \leq \frac{50}{100} + \frac{100 \times 5}{\sqrt{100 + 10000}} = 0.5 + \frac{100 \times 5}{\sqrt{10100}} \approx 5.5,$$

when $\alpha = 10$

$$|e_{ss}(t)| \leq \frac{50}{10} + \frac{10 \times 5}{\sqrt{100 + 100}} = 5 + \frac{5}{\sqrt{2}} \approx 8.5,$$

and when $\alpha = 1$

$$|e_{ss}(t)| \leq \frac{50}{1} + \frac{5}{\sqrt{100 + 1}} = 50 + \frac{1}{\sqrt{101}} \approx 50. \quad (+2.0 \text{ points})$$

Smallest error is when α is large. (+0.5 point)

There is no α for which error is less than $\bar{v}$ (+0.5 point)

Probably not :) Filter reduces v for α small but introduces a bias that can get much bigger than $\bar{v}$.

Total regular points: [5]

Bonus points: [2]

Your third design will be a second-order low-pass filter.

7. Estimator Design III

(a) [2 points] Let the estimator F be the second-order filter

$$F(s) = F_2(s) = \frac{\alpha\beta s + \alpha^2}{s^2 + \alpha\beta s + \alpha^2}, \quad \alpha > 0.$$

Show that

$$H(s) = H_2(s) = [1 - F_2(s)] G(s) = \frac{0.01s}{(s^2 + \alpha\beta s + \alpha^2)(s + 0.2)}.$$

Solution:

Compute

$$\begin{aligned}
 H(s) &= [1 - F(s)]G(s) \\
 &= \left[1 - \frac{\alpha\beta s + \alpha^2}{s^2 + \alpha\beta s + \alpha^2}\right] \frac{1}{s(ms + b)} \\
 &= \frac{s^2}{s^2 + \alpha\beta s + \alpha^2} \times \frac{1}{s(ms + b)} \\
 &= \frac{s(1/m)}{(s^2 + \alpha\beta s + \alpha^2)(s + b/m)}.
 \end{aligned}$$

(+2.0 points)

(b) [2 points (bonus)] Use Part (c) in Question 4 to show that

$$|x_{ss}(t) - \hat{x}_{ss}(t)| \leq \frac{\sqrt{1 + \lambda^2\beta^2/\alpha^2}}{\sqrt{(1 - \lambda^2/\alpha^2)^2 + \lambda^2\beta^2/\alpha^2}} |\bar{v}|.$$

What does $\bar{v} = 0$ mean in this formula?

Solution:

From Question 4.b)

$$|e_{ss}(t)| \leq |H(0)||\bar{r}| + |F(j\lambda)||\bar{v}|$$

In this case $H(0) = 0$ and

$$\begin{aligned}
 |F(j\lambda)| &= \left| \frac{j\alpha\beta\lambda + \alpha^2}{j\beta\alpha\lambda + (\alpha^2 - \lambda^2)} \right| \\
 &= \left| \frac{j\lambda\beta/\alpha + 1}{j\lambda\beta/\alpha + (1 - \lambda^2/\alpha^2)} \right| \\
 &= \frac{\sqrt{1 + \lambda^2\beta^2/\alpha^2}}{\sqrt{(1 - \lambda^2/\alpha^2)^2 + \lambda^2\beta^2/\alpha^2}}
 \end{aligned}$$

(+1.0 point)

When $\bar{v}$ the tracking error is zero, which means the estimate $\hat{x}$ converges to x when there is no measurement disturbance. (+1.0 point)

(c) [3 points] Use the formula in Part (b) to compute a bound for $|x_{ss}(t) - \bar{x}_{ss}(t)|$ when $\bar{v} = 5$ m, and $\lambda = 10$ rad/s, $\beta = 0.5$, and the values of α are

$$\alpha = 1, \quad \alpha = 10, \quad \alpha = 100.$$

What values of α lead to the smallest estimation error? Is there any value of α for which the estimation error is smaller than $\bar{v} = 5$ m? Is that what you expected?

Solution:

From Part (b)

$$|e_{ss}(t)| \leq \frac{\sqrt{1 + \lambda^2 \beta^2 / \alpha^2}}{\sqrt{(1 - \lambda^2 / \alpha^2)^2 + \lambda^2 \beta^2 / \alpha^2}} |\bar{v}|$$

When $\bar{v} = 5$, $\lambda = 10$, $\beta = 0.5$, and $\alpha = 100$

$$|e_{ss}(t)| \leq \frac{\sqrt{1 + 0.25/100}}{\sqrt{(1 - 1/100)^2 + 0.25/100}} 5 \approx 5$$

When $\alpha = 10$

$$|e_{ss}(t)| \leq \frac{\sqrt{1 + 0.25}}{\sqrt{0.25}} 5 \approx 11$$

When $\alpha = 1$

$$|e_{ss}(t)| \leq \frac{\sqrt{1 + 25}}{\sqrt{(1 - 100)^2 + 25}} 5 \approx 0.26 \quad (+2.0 \text{ points})$$

Smallest error is when α is small. (+1.0 point)

For $\alpha = 1$ the error is much smaller than $\bar{v}$. (+1.0 point)

Yes, this is the expected low pass characteristic! Bias was removed by the additional zero at $s = 0$.

- (d) [1 point (bonus)] Name one obstacle for making α too small.

Solution:

The error estimate is based on the steady-state solution. A smaller α means a slower transient as α appears as a pole (ω_n) in H_2 and in $F_2(s)$.

Total regular points: [5]

Bonus points: [3]

8. Bonus Question

- (a) [2 points (bonus)] Show that if $G(0) \neq 0$ and $F(0) \neq 1$ then $H(0) \neq 0$.

Solution:

If $F(0) \neq 1$ then $1 - F(0) \neq 0$. If $G(0) \neq 0$ then

$$H(0) = [1 - F(0)]G(0) \neq 0. \quad (+2.0 \text{ points})$$

- (b) [2 points (bonus)] Compute the zeros of $1 - F(s)$ for $F_1(s)$ and $F_2(s)$. Explain how zeros of $1 - F(s)$ at $s = 0$ affect $H(0)$.

Solution:

As computed before,

$$1 - F_1(s) = 1 - \frac{\alpha}{s + \alpha} = \frac{s}{s + \alpha}$$

which has a single zero at $s = 0$, and (+0.5 point)

$$1 - F_2(s) = 1 - \frac{\alpha\beta s + \alpha^2}{s^2 + \alpha\beta s + \alpha^2} = \frac{s^2}{s^2 + \alpha\beta s + \alpha^2}$$

which has a double zero at $s = 0$. (+0.5 point)

Because $G(s)$ has a pole at $s = 0$, the first zero of $1 - F(s)$ will cancel that pole. Hence, if m is the number of zeros of $G(s)$ at $s = 0$, then $H(s)$ will have $m - 1$ zeros at $s = 0$. If $m > 1$ then $H(0) = 0$. (+1.0 point)

- (c) [2 points (bonus)] In the free-fall model, if $b = 0$, what property should $F(s)$ have so that $H(0) = 0$? Do not forget that $F(s)$ has to be proper!

Solution:

If $b = 0$ then

$$G(s) = \frac{1}{ms^2},$$

has two poles at $s = 0$. (+1.0 point)

For them to be canceled and $H(0)$ to be zero we will need to have $1 - F(s)$ have three zeros at $s = 0$. That means $F(s)$ will have to be of order three or higher. (+1.0 point)

For example

$$F_3(s) = \frac{a_1 s^2 + a_2 s + a_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

is such that

$$1 - F_3(s) = 1 - \frac{a_1 s^2 + a_2 s + a_3}{s^3 + a_1 s^2 + a_2 s + a_3} = \frac{s^3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

has a triple zero at $s = 0$. (+1.0 point)

- (d) [2 points (bonus)] In Question 2, you have identified the frequency response of the filters F_1 and F_2 , and the transfer-functions H_1 and H_2 plotted in Figure 3 for $\alpha = 1, \beta = 0.5$. What impact on the estimation error would a measurement disturbance v with frequency $\lambda = 1$ rad/s have? Analyze both filters F_1 and F_2 based on the plots, no new calculations needed!

Solution:

The component of the error due to $H(0)\bar{r}$ would remain the same. However,

from the plots

$$F_1(j1) \approx -3\text{dB} \approx 0.7 \qquad F_2(j1) \approx +8\text{dB} \approx 2.5 \qquad (+1.0 \text{ point})$$

In this case, in the case of $F_1(s)$, $\alpha = 1$, $\bar{v} = 5$

$$|e_{ss}(t)| \leq 50 + 0.7 \times 5 = 50.35 \text{ m}$$

whereas

$$|e_{ss}(t)| \leq 2.5 \times 5 = 12.5 \text{ m}$$

for $F_2(s)$. (+1.0 point)

Compare this with 0.26 m obtained at $\lambda = 10$!

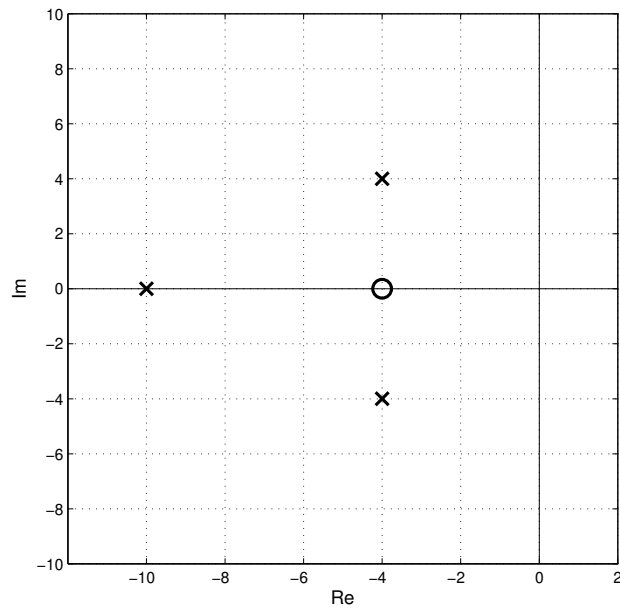
Bonus points: [8]

STUDENT NAME:

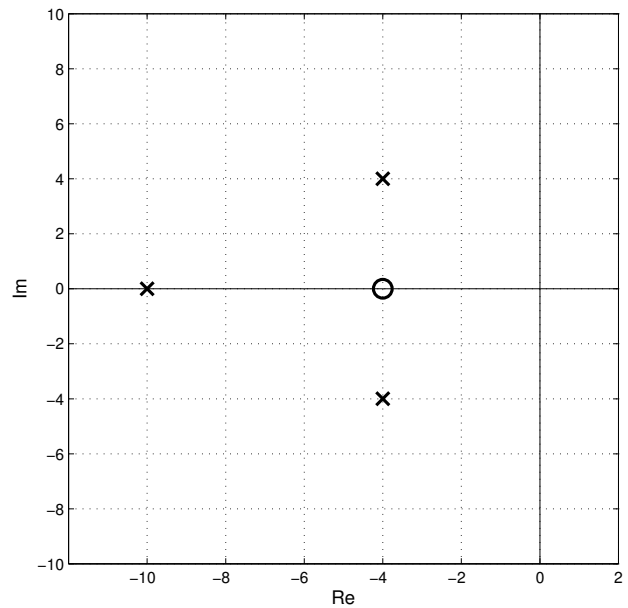
STUDENT ID:

WRITE YOUR NAME AND ID AND
DO NOT FORGET TO RETURN THIS PAGE WITH YOUR EXAM!

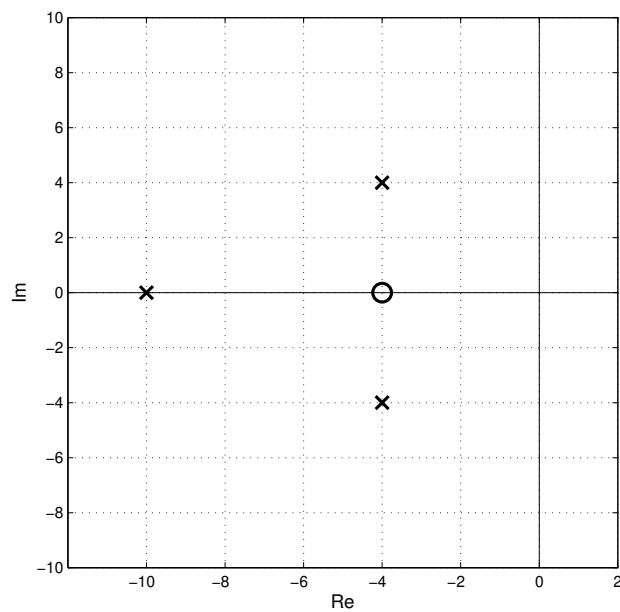
Figure 2: Pole-zero diagrams for Question 1



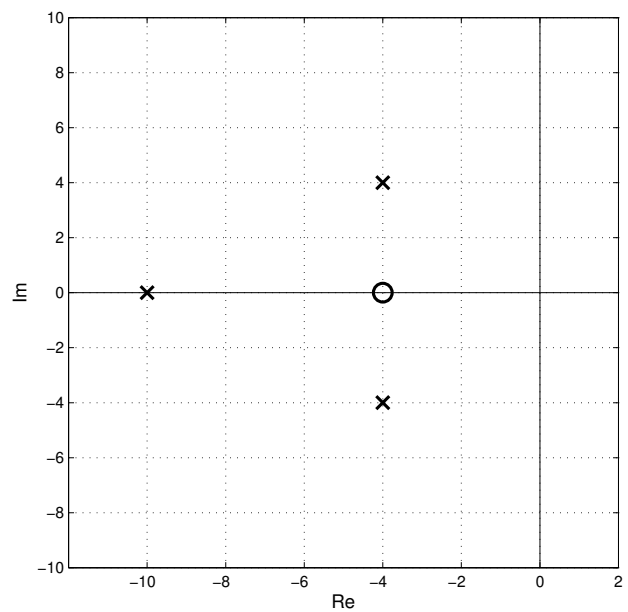
(a)



(b)



(c)



(d)

Figure 3: Bode plots for Question 2

