# MAE143b Hw 5 solutions

## Problem 9.1

a) Lets say that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

then $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix}$

Now we want the form $\dot{x} = f(x,u)$

so solve for $\ddot{\theta} \Rightarrow \ddot{\theta} = \dfrac{-(g\sin\theta + R\dot{\theta}^2)}{l + R\theta}$

$\dot{x}_2 = \ddot{\theta}$ and $\dot{x}_1 = \dot{\theta} = x_2$

so

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \dfrac{-(g\sin\theta + R\dot{\theta}^2)}{l + R\theta} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} x_2 \\ \dfrac{-(g\sin x_1 + R x_2^2)}{l + R x_1} \end{bmatrix}$$

Conti Prob 9.1

b) To get the linearized equations take the jacobian of $\dot{x} = f$

so $F = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 \\ \frac{Rg\sin(x_1) + R^2 x_2^2 - g\cos(x_1)(l + Rx_1)}{(l + Rx_1)^2} & -\frac{2Rx_2}{l + Rx} \end{bmatrix}$$

Now evaluate around $\theta = x_1 = 0$, $\dot{\theta} = x_2 = 0$

$$= \begin{bmatrix} 0 & 1 \\ \frac{-gl}{l^2} & 0 \end{bmatrix}$$

so

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

meaning

$$\ddot{\theta} = -\frac{g}{l}\theta \implies \ddot{\theta} + \frac{g}{l}\theta = 0$$

Conti Prob. 9.1

b) Another way to linearize:

Lets take $(l+R\theta)\ddot{\theta}+g\sin\theta+R\dot{\theta}^2=0$

For small $\theta$:

$\sin\theta \approx \theta$

$R\theta \approx 0$ plugging these in

$\dot{\theta}^2 \approx 0$

$l\ddot{\theta}+g\theta=0$ divide by $l$

$\ddot{\theta}+\frac{g\theta}{l}=0$ which is the Linearized equation

## Problem 9.2

In equilibrium $\frac{di}{dt}=-i+v=0$

$\frac{dv}{dt}=-i+g(u-v)=0$

Take $-i+g(u-v)=0$

$G=\frac{i_G}{V_g}$

$V_g=u-v$

$i_G=i$

so

$-v+(u-v)((u-v)-1)((u-v)-4)=0$

$=v(v^2+2v-2)=0$

$v=0,\ -1\pm\sqrt{3}$

now from $-i+v=0$

$i=v$ so

$i=v=0, -1\pm\sqrt{3}$

b) So at equilibrom point $u=1$ where $i=v=0$

replace $u$ $v$ and $i$ with $1+\delta u$, $\delta v$ and $\delta i$

so for $\dot{i}=-i+v \implies \delta\dot{i}=-\delta i+\delta v$

for $\dot{v}=-i+g(u-v)$

$\dot{v}=-i+(u-v)((u-v)-1)((u-v)-4)$

$$\delta\dot{v}=-\delta i+(1+\delta u-\delta v)((1+\delta u-\delta v)(1+\delta u-\delta v-4)$$
$$=-\delta i+(1+\delta u-\delta v)(\delta u-\delta v)(\delta u-\delta v-3)$$
$$\cong -\delta i-3\delta u+3\delta v$$

so $$\frac{d}{dt}\begin{bmatrix}\delta i\\ \delta v\end{bmatrix}=\begin{bmatrix}-1 & 1\\ -1 & 3\end{bmatrix}\begin{bmatrix}\delta i\\ \delta v\end{bmatrix}+\begin{bmatrix}0\\ -3\end{bmatrix}\delta u$$

c) In general the linearized form of the system will be :

$$\frac{d}{dt}\begin{bmatrix}\delta i\\ \delta v\end{bmatrix}=\begin{bmatrix}-1 & 1\\ -1 & \frac{\partial g}{\partial v}\end{bmatrix}\begin{bmatrix}\delta i\\ \delta v\end{bmatrix}+\begin{bmatrix}0\\ \frac{\partial g}{\partial u}\end{bmatrix}\delta u$$

$$g(v_g) = g(u,v) = (u-v)(u-v-1)(u-v-4)$$

for $u = 1$

$$g(1,v) = v(v^2+2v-2)$$

Now $\frac{\partial g}{v} = (v^2+2v-2) + v(2v+2)$

Now when $v = -1 \pm \sqrt{3}$

$$\frac{\partial g}{\partial v} = -5 - 2\sqrt{3}$$

when $v = -1-\sqrt{3}$

$$\frac{\partial g}{\partial v} = 5 + 2\sqrt{3}$$

Now consider $\frac{\partial g(u-v)}{\partial v} = -g'(u-v$

and $\frac{\partial g(u-v)}{\partial u} = g'(u-v) = -\frac{\partial g}{\partial v}$

so $\frac{\partial g}{\partial u} = -5 \pm 2\sqrt{3}$

so

$$\frac{d}{dt}\begin{bmatrix} \delta i \\ \partial v \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 5 \mp 2\sqrt{3} \end{bmatrix}\begin{bmatrix} \partial i \\ \partial v \end{bmatrix} + \begin{bmatrix} 0 \\ -5 \pm 2\sqrt{3} \end{bmatrix} \delta u$$