MAE143 B - Linear Control - Spring 2012
Midterm, May 8th

Instructions:

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities
2. You have 50 minutes. This is a relatively long exam for this amount of time. Do not attempt the bonus questions until you're done or unless you're sure of the answer
3. Most questions show you the answer, and you're asked to prove it. Use the given answers to move on to the next question. Do not get stuck!
4. Write your answers on your "Blue Book"
5. Do not forget to write your name, student number, and instructor

Questions:

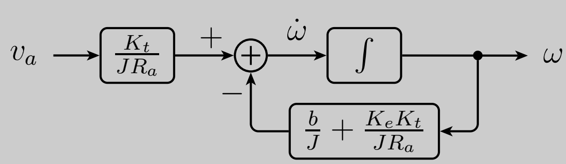
1. A DC motor  is modeled by the ordinary differential equation:

$$J\,\dot{\omega}(t) + \left(b + K_e \frac{K_t}{R_a}\right)\omega(t) = \frac{K_t}{R_a} v_a(t) \qquad \text{(SYS)}$$

where $\omega(t)$ is the angular velocity of the motor and $v_a(t)$ is the voltage applied. The constants J, b, and K_t are associated with mechanical parts of the motor and K_e and R_a are associated with electrical parts of the motor.

(a) (2 points) Draw a block diagram with integrators that represent the DC motor differential equation (SYS). Clearly identify the signals $\omega(t)$, $\dot{\omega}(t)$ and $v_a(t)$ in your diagram.

Solution:



(+2 points)

(b) (2 points) Show that the transfer function from $V_a(s)$ to $\Omega(s)$ is

$$G(s) = \frac{\Omega(s)}{V_a(s)} = \frac{\alpha}{s+\beta}$$

where

$$\alpha = \frac{K_t}{JR_a}, \qquad \beta = \left(\frac{b}{J} + \frac{K_e K_t}{JR_a}\right).$$

Use α and β as the motor parameters in the rest of this exam.

Solution:
Applying the Laplace Transform with zero initial conditions to (SYS) we obtain:

$$sJ\,\Omega(s) + \left(b + K_e\frac{K_t}{R_a}\right)\Omega(s) = \frac{K_t}{R_a}V_a(s). \qquad \text{(+1 point)}$$

Dividing by J we obtain

$$(s+\beta)\Omega(s) = \alpha V_a(s) \qquad \text{(+0.5 point)}$$

where α and β are as given. Dividing by $s+\beta$

$$\Omega(s) = \frac{\alpha}{s+\beta}V_a(s) = G(s)V_a(s). \qquad \text{(+0.5 point)}$$

(c) (2 points) The step response of the DC motor — i.e. $\omega(t)$ computed when $v_a(t)$ is a unit step, $1(t)$ — is given by

$$\omega_{\text{step}}(t) = \frac{\alpha}{\beta}(1 - e^{-\beta t})1(t).$$

The time-constant of a system, τ, as measured in a step response, is the time it takes the system to reach $1 - e^{-1} \approx 0.63$ of the steady-state value. Show that for the DC motor $\tau = \beta^{-1}$.

Solution:
From $\omega_{\text{step}}(t)$, the steady state value of $\omega(t)$ is equal to α/β. (+1 point)
Dividing $\omega_{\text{step}}(t)$ by α/β and equating to $1 - e^{-1}$ at $t = \tau$ we obtain

$$\frac{\beta}{\alpha}\omega_{\text{step}}(\tau) = 1 - e^{-\beta\tau} = 1 - e^{-1} \qquad \text{(+0.5 point)}$$

from where

$$\beta\tau = 1 \qquad \Longrightarrow \qquad \tau = \beta^{-1}. \qquad \text{(+0.5 point)}$$

(d) (2 points) Use the Final Value Theorem to compute the steady-state value of the unit step response of the DC motor.
Hint: make sure your answer matches $\lim_{t\to\infty} \omega_{\text{step}}(t)$

Solution:
If $v_a(t) = 1(t)$ then $V_a(s) = 1/s$ and

$$\Omega_{\text{step}}(s) = \frac{\alpha}{s(s+\beta)} \qquad \text{(+1 point)}$$

Applying the final value theorem

$$\lim_{t\to\infty} \omega_{\text{step}}(t) = \lim_{s\to 0} s\,\Omega_{\text{step}}(s) = \lim_{s\to 0} \frac{\alpha}{s+\beta} = \frac{\alpha}{\beta}. \qquad \text{(+1 point)}$$

(e) (2 points) Show that a DC motor that takes $\tau = 10$ s to reach 0.63 of its steady state 1200 rpm in response to a 12 V step voltage $v_a(t)$ has parameters

$$\alpha = 10 \text{ rpm / (V s)}, \qquad \beta = 0.1 \text{ s}^{-1}.$$

Hint: use answer to part (c)

Solution:
From part (c), $\tau = 10$ s is the time-constant and therefore

$$\beta = \tau^{-1} = 0.1 \text{ s}^{-1}. \qquad \text{(+1 point)}$$

The steady state value 1200 rpm is

$$\frac{\alpha}{\beta} 12 \text{ V} = 1200 \text{ rpm}, \qquad \text{(+0.5 point)}$$

hence

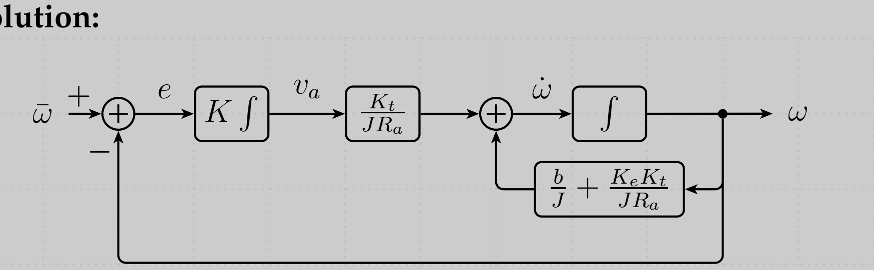
$$\alpha = \frac{1200}{12}\beta \text{ rpm/V} = 100 \times 0.1 \text{ rpm/(V s)} = 10 \text{ rpm/(V s)}. \qquad \text{(+0.5 point)}$$

2. Consider now the integral feedback controller

$$v_a(t) = K\int_0^t e(\tau)\,d\tau, \qquad e(t) = \bar{\omega}(t) - \omega(t), \qquad K > 0. \qquad \text{(CTRL)}$$

(a) (2 points) Draw a block diagram with integrators that represent the closed-loop connection of the controller (CTRL) with the DC motor differential equation (SYS). Clearly identify the signals $\omega(t)$, $\dot{\omega}(t)$, $\bar{\omega}(t)$, $e(t)$, and $v_a(t)$ in your diagram.

Solution:



(+2 points)

(b) (3 points) Show that the closed-loop transfer function from $\bar{\Omega}(s)$ to $\Omega(s)$ is of the form

$$H(s) = \frac{\Omega(s)}{\bar{\Omega}(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where

$$\omega_n = \sqrt{K\alpha}, \qquad \xi = \frac{\beta}{2\sqrt{K\alpha}}.$$

IMPORTANT: In the next parts, let $\alpha = 10$ and $\beta = 0.1$ as in question 1.(e).

Solution:
Taking the Laplace Transform of the controller

$$V_a(s) = \mathcal{L}\left\{K \int_0^t e(\tau)\, d\tau\right\} = K(s)E(s) \qquad E(s) = \bar{\Omega}(s) - \Omega(s) \qquad \text{(+0.5 point)}$$

where

$$K(s) = \frac{K}{s} \qquad \text{(+0.5 point)}$$

The closed-loop connection is standard with $G(s)$ from part 1.(b) and

$$H(s) = \frac{K(s)G(s)}{1 + K(s)G(s)} = \frac{\frac{K\alpha}{s(s+\beta)}}{1 + \frac{K\alpha}{s(s+\beta)}} = \frac{K\alpha}{s^2 + \beta s + K\alpha} \qquad \text{(+1 point)}$$

and ω_n and ξ come from the correspondence

$$\omega_n^2 = K\alpha, \qquad 2\xi\omega_n = b. \qquad \text{(+1 point)}$$

(c) (1 point) Compute the value of K for which $\omega_n = 1$ rad/s. What is the corresponding value of ξ?

Solution:
For $\omega_n = 1$ we need

$$\sqrt{K\alpha} = 1, \qquad \implies \qquad K = \alpha^{-1} = 0.1 \qquad \text{(+0.5 point)}$$

Therefore

$$\xi = \frac{\beta}{2\sqrt{K\alpha}} = \frac{0.1}{2\sqrt{0.1 \times 10}} = 0.05. \qquad \text{(+0.5 point)}$$

(d) (2 points) For which values of K is the closed-loop step-response from $\bar{\omega}(t)$ to $\omega(t)$ not oscillatory? Justify your answer.

Solution:

The response is not oscillatory if the poles of $H(s)$ are real. (+0.5 point)

For that to happen we need the discriminant

$$\Delta = 4\xi^2\omega_n^2 - 4\omega^2 = 4\omega_n^2(\xi^2 - 1) \qquad \text{(+0.5 point)}$$

to be nonnegative, or in other words

$$\xi^2 \geq 1 \qquad \Longrightarrow \qquad \xi \geq 1 \qquad \text{(+0.5 point)}$$

Hence

$$\xi = \frac{\beta}{2\sqrt{K\alpha}} \geq 1 \qquad \Longrightarrow \qquad K \leq \frac{\beta^2}{4\alpha} = \frac{0.01}{4 \times 10} = 25 \times 10^{-5}. \qquad \text{(+0.5 point)}$$

(e) (2 points) Compute the closed-loop steady-state value of $\omega(t)$ in response to a unit step, i.e. $\bar{\omega}(t) = 1(t)$. What is the steady-state tracking error, i.e. $\lim_{t\to\infty} e(t)$?

Solution:

The steady state value of the unit step response is given by

$$\lim_{t\to\infty} \omega(t) = \lim_{s\to 0} H(s) = \frac{K\alpha}{K\alpha} = 1 \qquad \text{(+1 point)}$$

and the steady state error is

$$\lim_{t\to\infty} \bar{\omega}(t) - \omega(t) = 1 - 1 = 0. \qquad \text{(+1 point)}$$

(f) (1 point (bonus)) Does the answer to part (e) depend on the value of $K > 0$? Explain why?

Solution:

It does not. The system is type-1 (contains a pole and zero), which ensures zero tracking error to step inputs as long as it is stable, which is the case for all $K > 0$.

(+1 point)

(g) (3 points (bonus)) Is it possible to select K such that $\omega_n \geq 1$ and ξ is at least 0.5? If not, can you modify your controller for that to be possible?

Solution:

For $\omega_n \geq 1$ we need

$$\omega_n = \sqrt{K\alpha} \geq 1 \qquad \Longrightarrow \qquad K \geq \alpha^{-1} = 0.1$$

and for $\xi \geq 0.5$ we need

$$\xi = \frac{\beta}{2\sqrt{K\alpha}} \geq 0.5 \qquad \Longrightarrow \qquad K \leq \frac{\beta^2}{\alpha} = \frac{0.01}{10} = 0.001$$

hence it is impossible. (+1 point)

Modifying the controller to be a PI controller

$$K(s) = K_p + \frac{K_i}{s} = K_p \frac{s + K_i/K_p}{s} \qquad \text{(+1 point)}$$

The closed-loop connection becomes

$$H(s) = \frac{K(s)G(s)}{1 + K(s)G(s)} = \frac{K_p \frac{(s+K_i/K_p)\alpha}{s(s+\beta)}}{1 + K_p \frac{(s+K_i/K_p)\alpha}{s(s+\beta)}} = \frac{K_p \alpha (s + K_i/K_p)}{s^2 + (\beta + K_p \alpha)s + K_i \alpha}$$

from which

$$\omega_n^2 = K_i \alpha, \qquad 2\xi\omega_n = \beta + K_p \alpha.$$

Therefore

$$\omega_n^2 = K_i \alpha \geq 1 \qquad \Longrightarrow \qquad K_i \geq \alpha^{-1} = 0.1$$

and

$$\xi = \frac{\beta + K_p \alpha}{2\omega_n} \geq 0.5 \qquad \Longrightarrow \qquad K_p \geq \frac{\omega_n - \beta}{\alpha} \geq \frac{1 - 0.1}{10} = 0.09.$$

Hence it is possible with a PI controller. (+1 point)