MAE143 B - Linear Control - Spring 2014
Final, June 10th

Instructions:

1. This exam is open book. You may use whatever written materials you choose, including your notes and the reader. You may use a calculator with no communication capabilities
2. You have 170 minutes
3. Do not get stuck! Move on and come back later if you have time
4. Write answers on your "Blue Book"
5. Do not forget to write your name, student number, and instructor
6. Initial conditions are zero unless otherwise noted
7. The exam has 6 question with for a total of 53 points and 9 bonus points
8. You may feel the urge to balance your pen or calculator, just don't break them :)

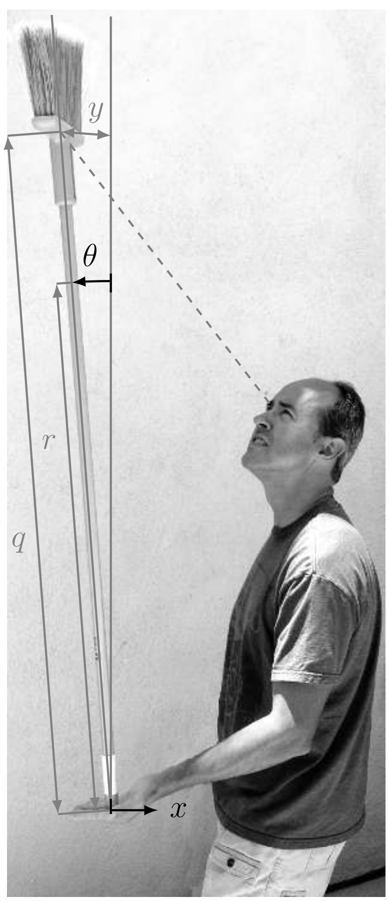

Figure 1: Your instructor balancing a broom

Questions:

1. **System Modeling.**

The nonlinear differential equation:

$$J_r\,\ddot{\theta} = m\,g\,r \sin\theta + m\,r\,\ddot{x}\cos\theta, \qquad\qquad J_r = J + m\,r^2,$$

is a simplified model for a broomstick being supported by a person's hand as shown in Figure 1, where θ and x are measured as indicated in the figure, r is the distance to the broom's center of mass, and m and J are the broom's mass and moment of inertia about its center of mass.

(a) (3 points) Let $u = \ddot{x}$ and draw a block-diagram relating the signals θ, $\dot{\theta}$, and u.

(b) (3 points) Write state-space equations based on the block diagram you obtained in part (a). Assume that $u = 0$ and calculate the only equilibrium point in the region $|\theta| < \pi/2$.

(c) (3 points) Let $y = -q\sin\theta$ be the difference between the x-component of the point your instructor is looking at in Figure 1 and his hand position. Show that

$$G_u(s) = \frac{Y(s)}{U(s)} = \frac{-q\,\sigma}{s^2 - g\,\sigma}, \qquad\qquad \sigma = \frac{m\,r}{J_r}$$

is the transfer-function associated with the state-space equations you obtained in part (b) linearized about $\theta = \dot{\theta} = u = 0$.

Hint: $\begin{bmatrix} s & a \\ b & s \end{bmatrix}^{-1} = \dfrac{1}{s^2 - ab}\begin{bmatrix} s & -a \\ -b & s \end{bmatrix}$.

(d) (2 points) Assume that all constants are positive. Is the transfer-function in part (c) asymptotically stable?

(e) (1 point (bonus)) What can you say about the equilibrium point of the original non-linear system?

(f) (1 point) Show that:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{-q\,\sigma\,s^2}{s^2 - g\,\sigma}$$

where x is the hand position. *Hint: Use (c).*

For the rest of the exam consider the data:

$$m = 1\text{kg}, \quad r = \frac{1}{2}\text{m}, \quad g = 10\text{m/s}^2, \quad J = \frac{1}{16}\text{kg m}^2, \quad q = 1\text{m}, \quad \sigma = \frac{16}{10}\text{m}^{-1}.$$

2. **Bode, Polar and Root-Locus Diagrams.**
   (a) (3 points) Sketch the magnitude and phase of the Bode plot of $G(j\omega)$.
   (b) (1 point (bonus)) Sketch the phase of the Bode plot of the minimum-phase transfer-function that has the same magnitude as $G(j\omega)$.
   (c) (2 points) Sketch the polar plot of $G(j\omega)$.
   (d) (3 points) Sketch the root-locus plot of the loop transfer-function $L(s) = -G(s)$. What can you say about balancing the broomstick using proportional control? *Hint: Don't get distracted by the minus sign. This is your standard root-locus plot!*

   In the next question you will transform the problem by removing the zeros from the model. This will make it simpler to design *a controller for the broomstick.*

3. **Problem Transformation.**
   (a) (2 points) Show that

   $$G(s) = \alpha\tilde{G}(s) + \alpha, \qquad \alpha = -q\,\sigma, \qquad \tilde{G}(s) = \alpha^{-1}G(s) - 1 = \frac{g\,\sigma}{s^2 - g\,\sigma}.$$

   (b) (3 points) Show that if the controller $\tilde{K}(s)$ asymptotically stabilizes the transfer-function $\tilde{G}(s)$ under unit negative feedback and no zero of $1 - \tilde{K}(s)$ is a zero of $1 + \tilde{K}(s)\tilde{G}(s)$ then the controller

   $$K(s) = \frac{\alpha^{-1}\tilde{K}(s)}{1 - \tilde{K}(s)} \tag{CTRL}$$

   asymptotically stabilizes the transfer-function $G(s)$ under unit negative feedback. *Hint: Show how $1 + \tilde{K}(s)\tilde{G}(s)$ is related to $1 + K(s)G(s)$.*

4. **Auxiliary Controller Design.**
   (a) (3 points) Sketch the root-locus plot for the loop transfer-function $L(s) = \tilde{G}(s)$. Can you asymptotically stabilize $\tilde{G}(s)$ using proportional control?
   (b) (5 points) Design a first-order controller with the structure:

   $$\tilde{K}(s) = K\frac{s+z}{s+p}$$

   that cancels the stable pole of $\tilde{G}(s)$ and select p and K so that the remaining closed-loop poles are both real and located at $s = -2$.
   (c) (3 points) Sketch the root-locus plot for the loop transfer-function $L(s) = \tilde{G}(s)\tilde{K}(s)$ with the $\tilde{K}(s)$ you computed in part (b). Explain your solution in terms of the root-locus method. Locate the closed-loop poles in the diagram. What is the root-locus gain that produces these closed-loop poles?
   (d) (3 points) Sketch the magnitude and phase of the Bode plot of the loop transfer-function $L(j\omega) = \tilde{G}(j\omega)\tilde{K}(j\omega)$.
   (e) (2 points (bonus)) Sketch the polar plot of the transfer-function $L(j\omega) = \tilde{G}(j\omega)\tilde{K}(j\omega)$ and explain why the controller $\tilde{K}(s)$ asymptotically stabilizes $\tilde{G}(s)$ using the Nyquist stability criterion.

5. **Final Controller.**

   (a) (3 points) The controller $K(s)$, equation (CTRL), can be interpreted as a feedback loop involving α and $\tilde{K}(s)$. Draw a block diagram showing this feedback loop.

   (b) (2 points (bonus)) Using the block diagram from part (a), draw a block diagram of the complete feedback loop that asymptotically stabilizes $G(s)$. Can you identify $\tilde{G}(s)$ in this diagram?

   (c) (3 points) Use (CTRL) to calculate a controller $K(s)$ that asymptotically stabilizes the broomstick based on the controller $\tilde{K}(s)$ you calculated in Q.4 part (b). Verify that the closed-loop poles are the same as those obtained in Q.4 part (c).
   Hint: You can verify that they are the same without recalculating the poles.

   (d) (3 points) Sketch the magnitude and phase of the Bode plot of the loop transfer-function $L(j\omega) = G(j\omega)K(j\omega)$.

   (e) (2 points (bonus)) Sketch the polar plot of the transfer-function $L(j\omega) = G(j\omega)K(j\omega)$ and explain why the controller $K(s)$ asymptotically stabilizes $G(s)$ using the Nyquist stability criterion.

6. **Reflections.**

   (a) (3 points) Why is your instructor looking up in Figure 1? What changes if he balances the broomstick by looking at the middle instead of at the top? Do you expect this to be easier or harder? What feature in your model explains your reasoning? Can he balance the broomstick by looking at his hand instead?

   (b) (2 points) Fix the cross section area and general shape of the broomstick. The mass of the broomstick, m, grows linearly with the length of the broomstick while the moment of inertia, J, grows linearly with the mass and quadratically with the length of the broomstick. Based on this observation, name one reason why it might be easier to control longer broomsticks.

   (c) (1 point (bonus)) Continuing part (b), name one reason why it might be harder to control longer broomsticks.