MAE143 B - Linear Control - Spring 2014
Final, June 10th

Instructions:

1. This exam is open book. You may use whatever written materials you choose, including your notes and the reader. You may use a calculator with no communication capabilities
2. You have 170 minutes
3. Do not get stuck! Move on and come back later if you have time
4. Write answers on your "Blue Book"
5. Do not forget to write your name, student number, and instructor
6. Initial conditions are zero unless otherwise noted
7. The exam has 6 question with for a total of 53 points and 9 bonus points
8. You may feel the urge to balance your pen or calculator, just don't break them :)

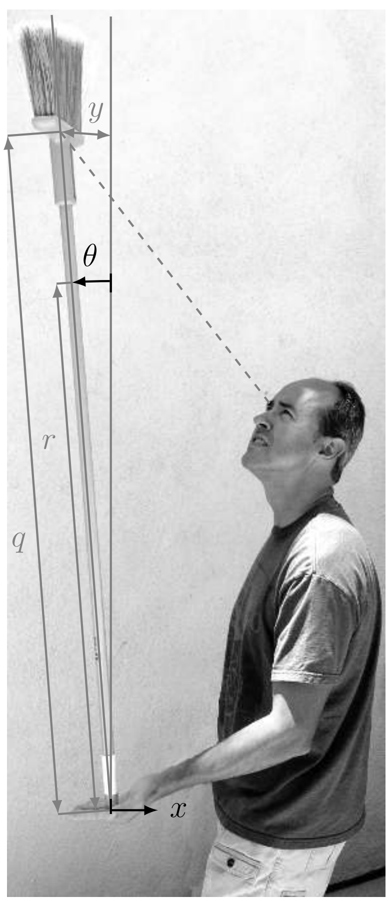


Figure 1: Your instructor balancing a broom

Questions:

1. **System Modeling.**

   The nonlinear differential equation:

$$J_r\,\ddot\theta = m\,g\,r\sin\theta + m\,r\,\ddot x\cos\theta, \qquad\qquad J_r = J + m\,r^2,$$

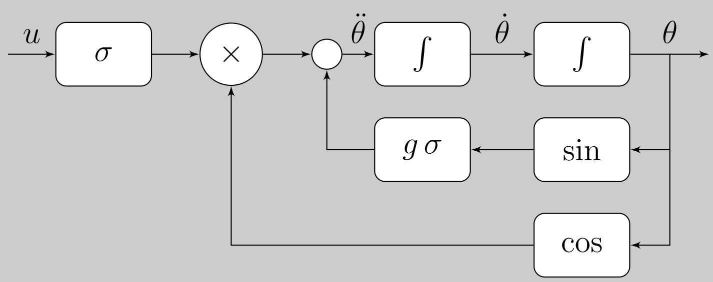
   is a simplified model for a broomstick being supported by a person's hand as shown in Figure 1, where θ and x are measured as indicated in the figure, r is the distance to the broom's center of mass, and m and J are the broom's mass and moment of inertia about its center of mass.

   (a) (3 points) Let $u = \ddot x$ and draw a block-diagram relating the signals θ, $\dot\theta$, and u.

   Solution:
   To obtain a block-diagram we rewrite:

$$\ddot\theta = g\,\sigma\sin\theta + \sigma u\cos\theta, \qquad\qquad \sigma = \frac{mr}{J_r}. \qquad (+1 \text{ point})$$

   from which we obtain the block diagram:



   (+1 point)

   NOTE TO GRADERS: You get the 1 point above if you get the integrators and the linear part right. You get an additional point if you get the entire block diagram right. (+1 point)

   (b) (3 points) Write state-space equations based on the block diagram you obtained in part (a). Assume that $u = 0$ and calculate the only equilibrium point in the region $|\theta| < \pi/2$.

   Solution:
   From part (a) define

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \theta \\ \dot\theta \end{pmatrix} \qquad (+1 \text{ point})$$

   and

$$\dot x = f(x,u) = \begin{pmatrix} x_2 \\ g\,\sigma\sin x_1 + \sigma u\cos x_1 \end{pmatrix}. \qquad (+1 \text{ point})$$

   At equilibrium with $\bar u = 0$ we need to have $f(\bar x, \bar u) = 0$. That is:

$$\bar x_2 = \dot\theta = 0, \qquad g\,\sigma\sin\bar x_1 = 0 \quad \Longrightarrow \quad \bar x_1 = \theta = 0 \qquad (+1 \text{ point})$$

   in the region $|\theta| < \pi/2$.

(c) (3 points) Let $y = -q \sin\theta$ be the difference between the x-component of the point your instructor is looking at in Figure 1 and his hand position. Show that

$$G_u(s) = \frac{Y(s)}{U(s)} = \frac{-q\,\sigma}{s^2 - g\,\sigma}, \qquad \sigma = \frac{m\,r}{J_r}$$

is the transfer-function associated with the state-space equations you obtained in part (b) linearized about $\theta = \dot\theta = u = 0$.

Hint: $\begin{bmatrix} s & a \\ b & s \end{bmatrix}^{-1} = \dfrac{1}{s^2 - ab}\begin{bmatrix} s & -a \\ -b & s \end{bmatrix}$.

Solution:
Define

$$y = \phi(x, u) = -q \sin x_1$$

and linearize to obtain:

$$\begin{aligned} \dot x &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{\bar x = \bar u = 0} = \begin{bmatrix} 0 & 1 \\ g\,\sigma\cos\bar x_1 - \sigma\bar u \sin\bar x_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ g\,\sigma & 0 \end{bmatrix} \qquad \text{(+1/2 point)}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{\bar x = \bar u = 0} = \begin{bmatrix} 0 \\ \sigma\cos\bar x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma \end{bmatrix} \qquad \text{(+1/2 point)}$$

$$C = \left.\frac{\partial \phi}{\partial x}\right|_{\bar x = \bar u = 0} = \begin{bmatrix} -q\cos\bar x_1 & 0 \end{bmatrix} = \begin{bmatrix} -q & 0 \end{bmatrix} \qquad \text{(+1/2 point)}$$

$$D = \left.\frac{\partial \phi}{\partial u}\right|_{\bar x = \bar u = 0} = \begin{bmatrix} 0 \end{bmatrix} \qquad \text{(+1/2 point)}$$

and calculate

$$\begin{aligned} G_u(s) &= C(sI - A)^{-1}B \\ &= \begin{bmatrix} -q & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ -g\,\sigma & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \sigma \end{bmatrix} \\ &= \frac{1}{s^2 - g\,\sigma} \begin{bmatrix} -q & 0 \end{bmatrix} \begin{bmatrix} s & 1 \\ g\,\sigma & s \end{bmatrix} \begin{bmatrix} 0 \\ \sigma \end{bmatrix} = \frac{-q\,\sigma}{s^2 - g\,\sigma} \end{aligned} \qquad \text{(+1 point)}$$

using the formula provided.

(d) (2 points) Assume that all constants are positive. Is the transfer-function in part (c) asymptotically stable?

Solution: If all constants are positive then the poles of the transfer-function are the roots of the polynomial:

$$s^2 - g\,\sigma = 0 \quad \Longrightarrow \quad s = \pm\sqrt{g\,\sigma}. \qquad \text{(+1 point)}$$

One of the roots is necessarily positive therefore the transfer-function is not asymptotically stable. (+1 point)

(e) (1 point (bonus)) What can you say about the equilibrium point of the original nonlinear system?

Solution: From Lyapunov Theorem (Lemma 5.1 in the reader), the original nonlinear system is unstable because one pole of the linearized system has positive real part. (+1 point)

(f) (1 point) Show that:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{-q\,\sigma\,s^2}{s^2 - g\,\sigma}$$

where x is the hand position. *Hint: Use (c).*

Solution: Because $u = \ddot{x}$ and

$$\frac{U(s)}{X(s)} = s^2 \qquad \text{(+1/2 point)}$$

we have that

$$G(s) = \frac{Y(s)}{X(s)} = \frac{Y(s)}{U(s)}\frac{U(s)}{X(s)} = s^2 G_u(s) = \frac{-q\,\sigma\,s^2}{s^2 - g\,\sigma} \qquad \text{(+1/2 point)}$$

For the rest of the exam consider the data:

$$m = 1\text{kg}, \quad r = \frac{1}{2}\text{m}, \quad g = 10\text{m/s}^2, \quad J = \frac{1}{16}\text{kg}\,\text{m}^2, \quad q = 1\text{m}, \quad \sigma = \frac{16}{10}\text{m}^{-1}.$$

2. **Bode, Polar and Root-Locus Diagrams.**

(a) (3 points) Sketch the magnitude and phase of the Bode plot of $G(j\omega)$.

Solution:

Evaluate the constants to obtain:

$$G(s) = \frac{-1.6s^2}{s^2 - 16}$$

and normalize:

$$G(s) = \frac{1.6}{16}\frac{s^2}{1-(s/4)^2} = 0.1\frac{s^2}{1-(s/4)^2}. \qquad \text{(+1 point)}$$

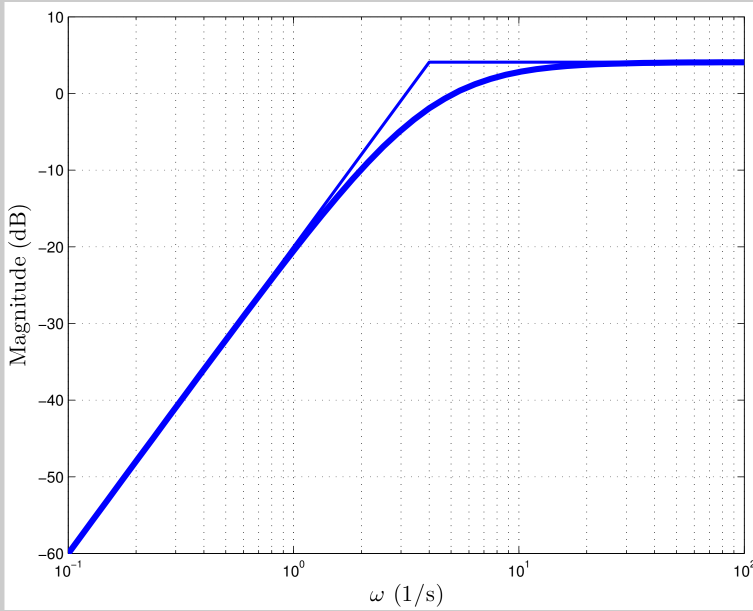
In the magnitude plot there will be a line of constant slope $+40$db/decade starting at $\omega \to 0$ and the slope will drop to 0db/decade after the double pole at $\omega = 4$. The straight-line approximations meet at $\omega = 4$ at

$$0.1|(j4)^2| = 0.1|(j4)^2| = 1.6 \qquad \text{(+1/2 point)}$$

In dBs:

$$20 \log_{10} 1.6 = 20(\log_{10} 16 - 1) \approx 20(0.2) \approx 4. \qquad \text{(+1/2 point)}$$

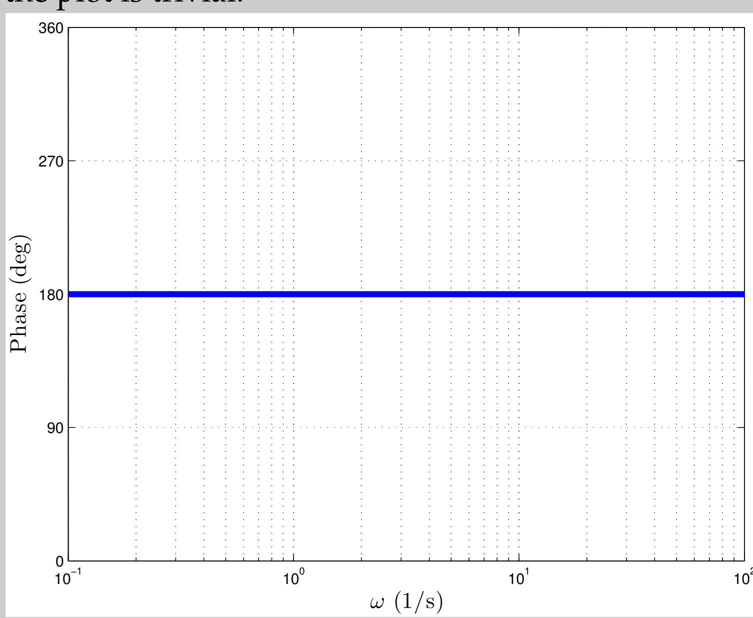
which leads to the plot:



The phase

$$\begin{aligned} \angle G(j\omega) &= \angle 0.1 + \angle -\omega^2 - \angle 1 + (\omega/4)^2 \\ &= 0 + \pi + 0 = \pi \end{aligned} \qquad \text{(+1 point)}$$

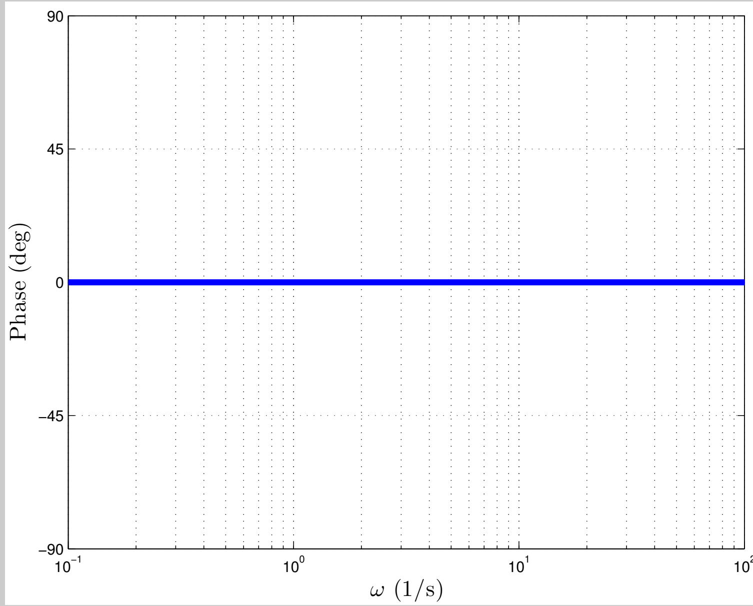
is constant so the plot is trivial:



(b) (1 point (bonus)) Sketch the phase of the Bode plot of the minimum-phase transfer-function that has the same magnitude as $G(j\omega)$.

Solution:

From part (a), the phase of $G(j\omega) = \pi$ for all ω. The corresponding minimum phase transfer-function will have $G(j\omega) = 0$ for all ω. The plot is trivial:



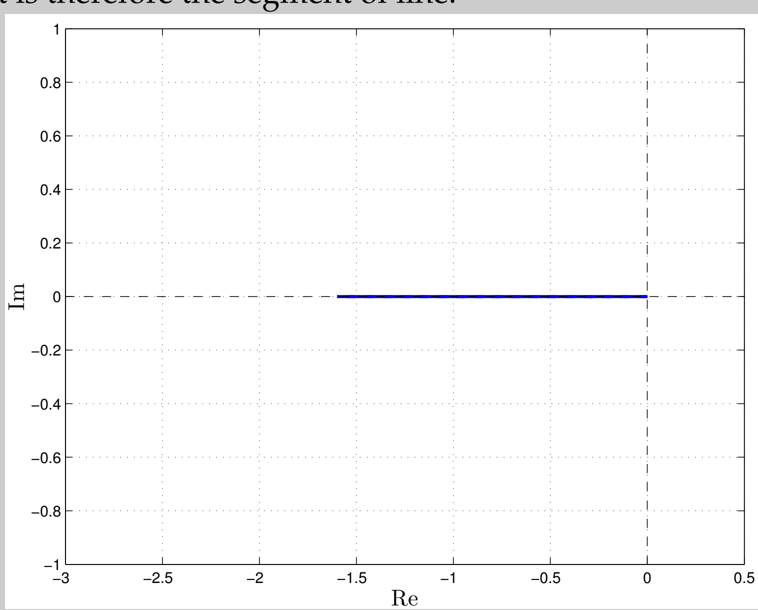
(+1 point)

(c) (2 points) Sketch the polar plot of $G(j\omega)$.

Solution:

Because the phase is constant and equal to π the polar plot is on the negative real axis. From the Bode plot in part (a) it starts at 0 at $\omega \to 0^+$ and grows to have magnitude 1.6 when $\omega \to \infty$. (+1/2 point)

The polar plot is therefore the segment of line:



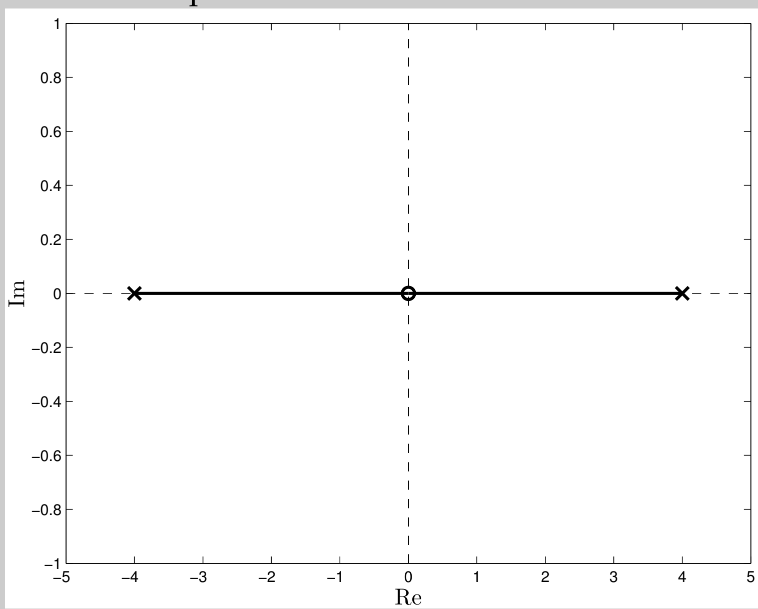
(+1 point)

The part corresponding to $\omega < 0$ also coincides with that line, starting at -1.6 at $\omega \to -\infty$ and growing to 0 as $\omega \to 0^-$. (+1/2 point)

(d) (3 points) Sketch the root-locus plot of the loop transfer-function $L(s) = -G(s)$.

What can you say about balancing the broomstick using proportional control?
Hint: Don't get distracted by the minus sign. This is your standard root-locus plot!

Solution:
The root-locus has no asymptotes because the number of poles is equal to the number of zeros. (+1 point)
The plot remains on the real axis, each pole going to a zero at the origin, following the standard rules. (+1 point)
The result is the root-locus plot:



(+1 point)

In the next question you will transform the problem by removing the zeros from the model. This will make it simpler to design *a controller for the broomstick.*

3. **Problem Transformation.**
   (a) (2 points) Show that

$$G(s) = \alpha\tilde{G}(s) + \alpha, \qquad \alpha = -q\,\sigma, \qquad \tilde{G}(s) = \alpha^{-1}G(s) - 1 = \frac{g\,\sigma}{s^2 - g\,\sigma}.$$

Solution:
You can show it in one of two ways.
Evaluate $\tilde{G}$:

$$\tilde{G}(s) = 1 - \alpha^{-1}\frac{-q\sigma s^2}{s^2 - g\sigma} = 1 - \frac{s^2}{s^2 - g\sigma} = \frac{s^2 - g\sigma - s^2}{s^2 - g\sigma} = \frac{g\sigma}{s^2 - g\sigma}. \qquad \text{(+2 points)}$$

OR
Evaluate G:

$$G(s) = \alpha\tilde{G}(s) + \alpha = \alpha\frac{g\,\sigma}{s^2 - g\,\sigma} + \alpha = \alpha\frac{g\,\sigma + s^2 - g\,\sigma}{s^2 - g\,\sigma} = \frac{\alpha s^2}{s^2 - g\,\sigma} = \frac{-q\sigma s^2}{s^2 - g\,\sigma}$$

(+2 points)

(b) (3 points) Show that if the controller $\tilde{K}(s)$ asymptotically stabilizes the transfer-function $\tilde{G}(s)$ under unit negative feedback and no zero of $1 - \tilde{K}(s)$ is a zero of $1 + \tilde{K}(s)\tilde{G}(s)$ then the controller

$$K(s) = \frac{\alpha^{-1}\tilde{K}(s)}{1 - \tilde{K}(s)} \tag{CTRL}$$

asymptotically stabilizes the transfer-function $G(s)$ under unit negative feedback. *Hint: Show how* $1 + \tilde{K}(s)\tilde{G}(s)$ *is related to* $1 + K(s)G(s)$.

Solution:
Follow the hint to obtain:

$$\begin{aligned} 1 + \tilde{K}(s)\tilde{G}(s) &= 1 + \tilde{K}(s)[\alpha^{-1}G(s) - 1] \\ &= 1 - \tilde{K}(s) + \alpha^{-1}\tilde{K}(s)G(s) \\ &= (1 - \tilde{K}(s))\left[1 + \frac{\alpha^{-1}\tilde{K}(s)}{1 - \tilde{K}(s)}G(s)\right] \\ &= (1 - \tilde{K}(s))\left[1 + K(s)G(s)\right] && \text{(+2 points)} \end{aligned}$$

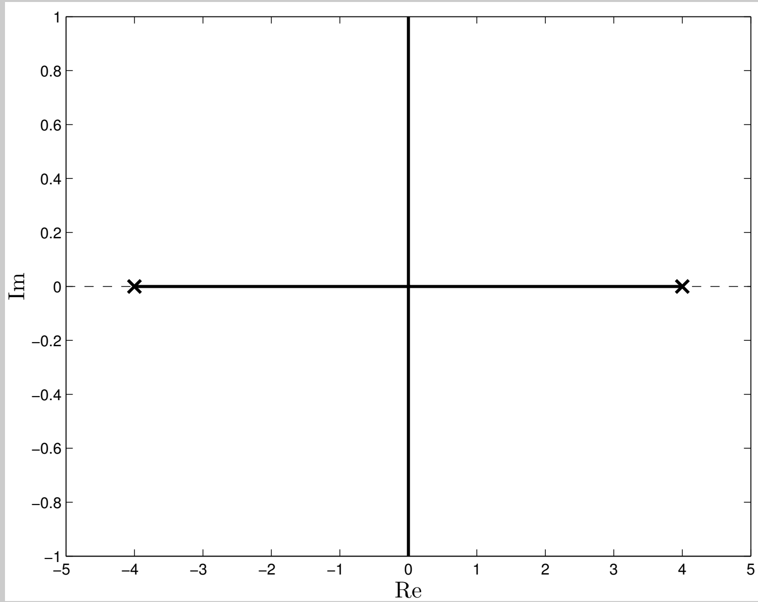
If s is a zero of $1 + \tilde{K}\tilde{G}$ and not a zero of $1 - \tilde{K}$ then it must also be a zero of $1 + KG$. (+1 point)

4. **Auxiliary Controller Design.**

(a) (3 points) Sketch the root-locus plot for the loop transfer-function $L(s) = \tilde{G}(s)$. Can you asymptotically stabilize $\tilde{G}(s)$ using proportional control?

Solution:
The root-locus has two asymptotes because the number of poles is equal to two and there are no zeros. They asymptotes meet at the origin because the poles are symmetric. In other words, they coincide with the imaginary axis. (+1 point)

From both poles, the locus remain real until reaching the origin and becoming imaginary. The root-locus plot is:



(+1 point)

One cannot stabilize with proportional control because, from the root-locus, for all positive gain $\alpha > 0$ there will be either a pole with positive real part or a pair of imaginary poles. (+1 point)

(b) (5 points) Design a first-order controller with the structure:

$$\tilde{K}(s) = K\frac{s+z}{s+p}$$

that cancels the stable pole of $\tilde{G}(s)$ and select p and K so that the remaining closed-loop poles are both real and located at $s = -2$.

Solution:

Choose $z = 4$ to cancel the stable pole of $\tilde{G}(s)$ so that:

$$\tilde{K}(s)\tilde{G}(s) = K\frac{s+4}{s+p} \times \frac{16}{s^2-16} = \frac{16K}{(s+p)(s-4)} \qquad \text{(+1 point)}$$

The closed-loop poles are the zeros of:

$$\begin{aligned} 1 + \tilde{K}(s)\tilde{G}(s) &= 1 + \frac{16K}{(s+p)(s-4)} \\ &= \frac{(s+p)(s-4) + 16K}{(s+p)(s-4)} \\ &= \frac{s^2 + (p-4)s + (16K - 4p)}{(s+p)(s-4)}. \end{aligned} \qquad \text{(+1 point)}$$

We seek for K and p such that

$$s^2 + (p-4)s + (16K - 4p) = (s+2)^2 = s^2 + 4s + 4. \qquad \text{(+1 point)}$$

For that to happen:

$$p - 4 = 4, \qquad 16K - 4p = 4$$

which has as solution:

$$p = 8, \qquad K = \frac{36}{16} = \frac{9}{4} = 2.25. \qquad \text{(+1 point)}$$

The resulting $\tilde{K}(s)$ is

$$\tilde{K}(s) = \frac{9}{4}\frac{s+4}{s+8}. \qquad \text{(+1 point)}$$

(c) (3 points) Sketch the root-locus plot for the loop transfer-function $L(s) = \tilde{G}(s)\tilde{K}(s)$ with the $\tilde{K}(s)$ you computed in part (b). Explain your solution in terms of the root-locus method. Locate the closed-loop poles in the diagram. What is the root-locus gain that produces these closed-loop poles?

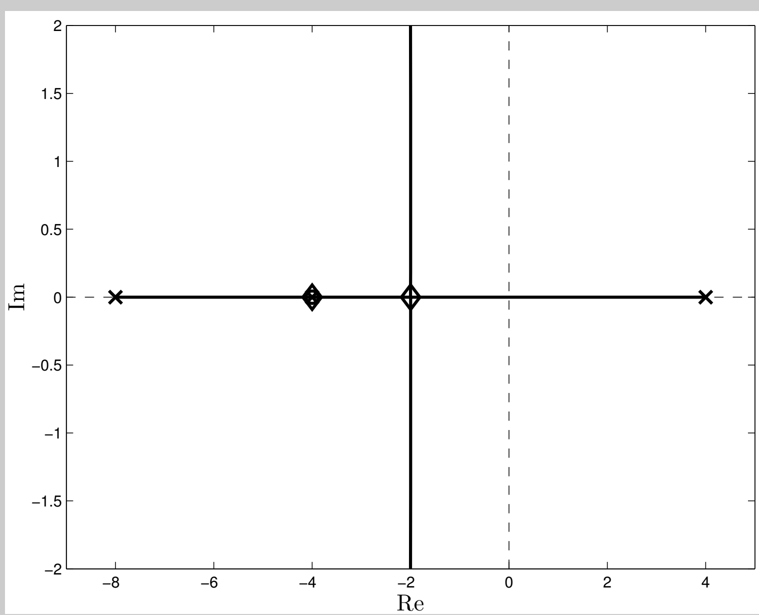
Solution:
From part (b):

$$L(s) = \tilde{K}(s)\tilde{G}(s) = \frac{36}{(s+8)(s-4)}$$

Because of the pole-zero cancellation the root-locus has two asymptotes because the number of poles is equal to two and there are no zeros. They asymptotes meet at:

$$\frac{-8+4}{2} = -2. \qquad \text{(+1 point)}$$

The value of K selected in part (b) places two closed-loop poles exactly at -2, which is were the asymptotes intersect. They are marked with a diamond in the root-locus plot:



(+1 point)

The root-locus gain that correspond to these poles is $\alpha = 1$ because we know from part (b) that they are roots of $1 + \alpha\tilde{K}(s)\tilde{G}(s) = 0$, $\alpha = 1$. (+1 point)

NOTE TO GRADERS: If the student correctly identifies the canceled pole $s = -4$ as one of the closed-loop poles then add an extra bonus point. (+1 bonus point)

(d) (3 points) Sketch the magnitude and phase of the Bode plot of the loop transfer-function $L(j\omega) = \tilde{G}(j\omega)\tilde{K}(j\omega)$.

Solution:

From part (b):

$$L(s) = \tilde{K}(s)\tilde{G}(s) = \frac{36}{(s+8)(s-4)}$$

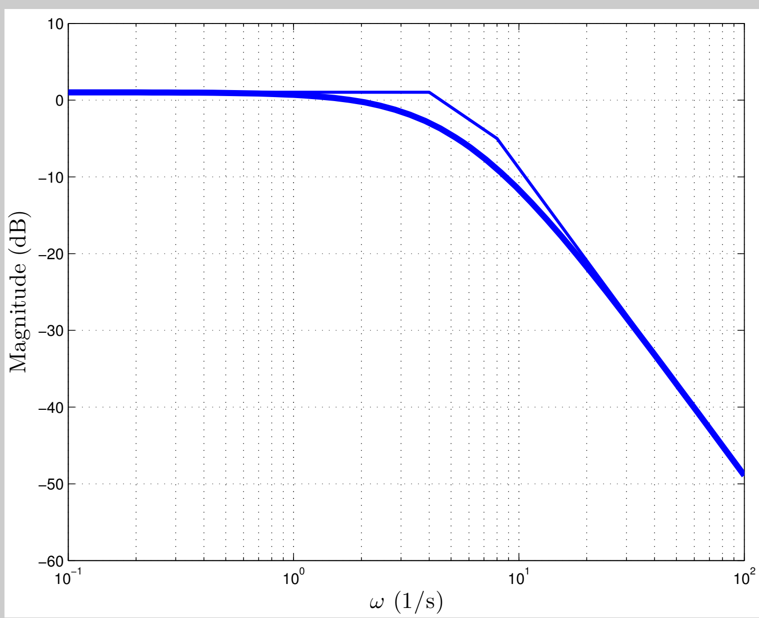
which we normalize:

$$L(s) = \frac{36}{8 \times -4}\frac{1}{(1+s/8)(1-s/4)} = \frac{-9}{8}\frac{1}{(1+s/8)(1-s/4)}$$

In the magnitude plot there will be a line of slope 0 starting at

$$20\log_{10} 9/8 \approx 20 \times 0.05 = 1\text{dB}. \qquad \text{(+1/2 point)}$$

The slope drops to -20dB/decade at $\omega = 4$ and then to -40dB/decade at $\omega = 8$. The Bode plot looks like:



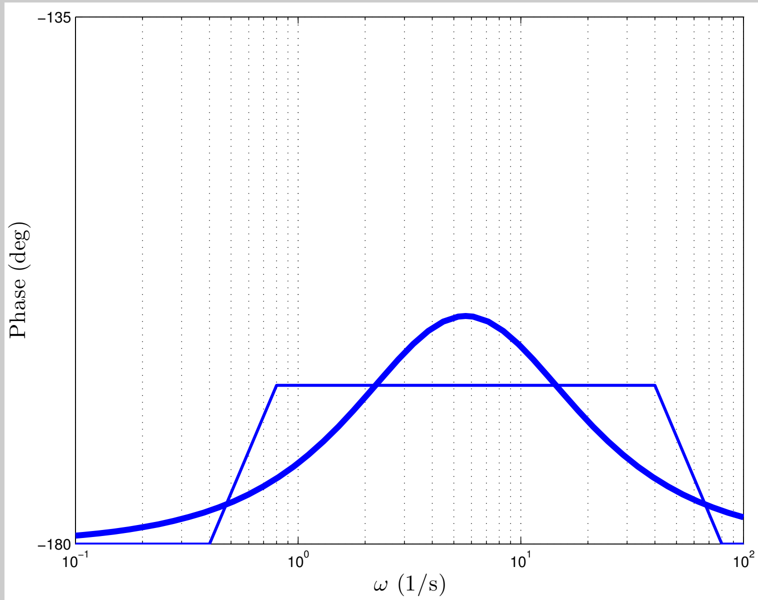
(+1 point)

The phase

$$\angle L(s) = \angle\frac{-9}{8} - \angle(1 + j\omega/8) - \angle(1 - j\omega/4)$$

so the plot starts at $-180°$ (or whatever even multiple you like) then has a positive slope of $45°$/decade starting at $\omega = 0.4$ due to the pole at $s = 4$ with positive real part. The slope cancels out at $\omega = 0.8$ due to the pole at $s = 8$ with negative real part. The slope remains at zero until $\omega = 40$ at which point it becomes $-45°$/decade and then back to zero at $\omega = 80$. (+1/2 point)

The plot looks like:



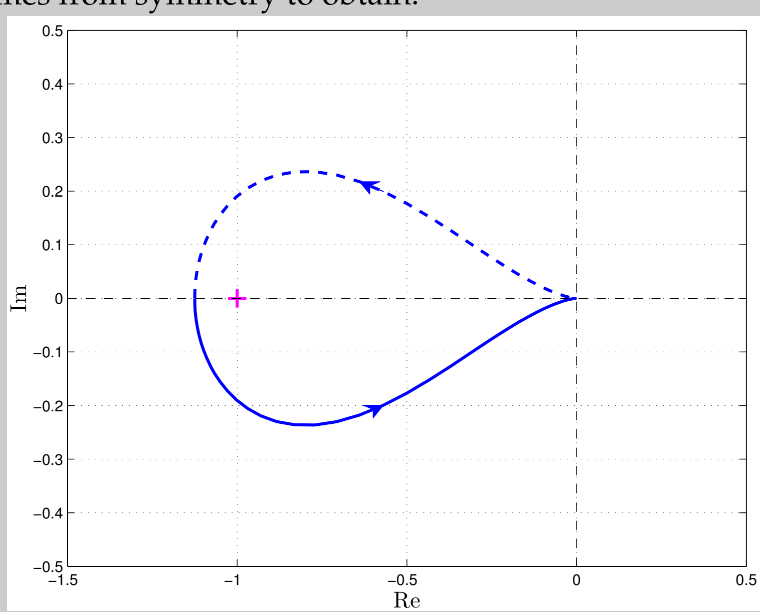
(+1 point)

(e) (2 points (bonus)) Sketch the polar plot of the transfer-function $L(j\omega) = \tilde{G}(j\omega)\tilde{K}(j\omega)$ and explain why the controller $\tilde{K}(s)$ asymptotically stabilizes $\tilde{G}(s)$ using the Nyquist stability criterion.

Solution:

A sketch of the polar plot of $L(s) = \tilde{K}(s)\tilde{G}(s)$ can be obtained from the Bode plot in part (d).

It starts at the point $-9/8$ from where the phase grows while the magnitude decreases until the phase starts decreasing again to reach $-180°$ at $\omega \to \infty$. The rest of the plot comes from symmetry to obtain:



(+1 point)

L has one pole on the right-hand side of the complex plane so that $Z_\gamma = 1$. The polar plot encloses the point -1 once in the clockwise direction so that

$$Z_\Gamma = P_\Gamma + \frac{1}{2\pi}\Delta_\Gamma^{-1/\alpha} \arg L(s) = 1 - 1 = 0$$

and the conclusion is that the closed-loop system is asymptotically stable according to the Nyquist stability criterion.

5. **Final Controller.**

(a) (3 points) The controller $K(s)$, equation (CTRL), can be interpreted as a feedback loop involving α and $\tilde{K}(s)$. Draw a block diagram showing this feedback loop.

Solution:

Many possible ways to see a solution here. I will show one. The formula is:

$$K(s) = \frac{\alpha^{-1}\tilde{K}(s)}{1 - \tilde{K}(s)}$$

which we can rewrite as:

$$K(s) = \alpha^{-1}\frac{\tilde{K}(s)}{1 + (-1)\tilde{K}(s)} \qquad \text{(+1 point)}$$

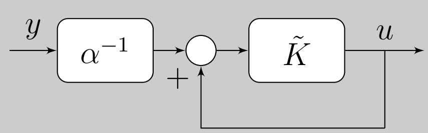
which is the same as positive feedback loop, because of the -1 gain, around $\tilde{K}(s)$ multiplied by α^{-1}. (+1 point)

OR

Another way to see it is if $K(s) = U(s)/Y(s)$ then

$$(1 - \tilde{K})u = \alpha^{-1}\tilde{K}y \qquad \Longrightarrow \qquad u = \tilde{K}(\alpha^{-1}y + u) \qquad \text{(+2 points)}$$

Either way the block diagram is:



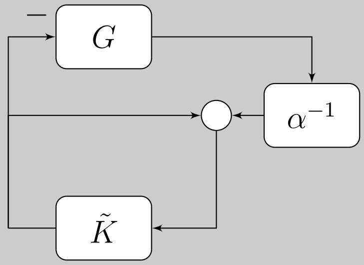
(+1 point)

NOTE TO GRADER: Alternative diagrams exist in which α appears inside the loop, etc. Those are correct as well. Please verify.

(b) (2 points (bonus)) Using the block diagram from part (a), draw a block diagram of the complete feedback loop that asymptotically stabilizes $G(s)$. Can you identify $\tilde{G}(s)$ in this diagram?

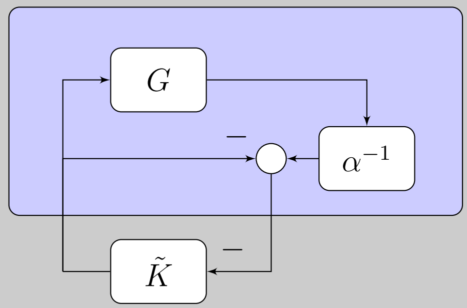
Solution:

We put K in negative feedback with G to obtain the block diagram:



(+1 point)

We shuffle the negative sign to reveal $\tilde{G}$ is inside the shaded rectangle.



(+1 point)

(c) (3 points) Use (CTRL) to calculate a controller $K(s)$ that asymptotically stabilizes the broomstick based on the controller $\tilde{K}(s)$ you calculated in Q.4 part (b). Verify that the closed-loop poles are the same as those obtained in Q.4 part (c).
Hint: You can verify that they are the same without recalculating the poles.

Solution:

From Question 4(b)

$$\tilde{K}(s) = \frac{9}{4}\frac{s+4}{s+8}, \qquad \alpha = -q\sigma = -\frac{16}{10}.$$

We first calculate the controller:

$$\begin{aligned} K(s) &= \frac{\alpha^{-1}\tilde{K}(s)}{1-\tilde{K}(s)} \\ &= -\frac{10}{16}\frac{\frac{9}{4}\frac{s+4}{s+8}}{1-\frac{9}{4}\frac{s+4}{s+8}} \\ &= -\frac{10}{16}\frac{9(s+4)}{4(s+8)-9(s+4)} \\ &= -\frac{10}{16}\frac{9(s+4)}{-(5s+4)} \\ &= \frac{9}{8}\frac{s+4}{s+4/5} \end{aligned}$$

(+1 point)

and

$$\begin{aligned} K(s)\,G(s) &= \frac{9}{8}\frac{s+4}{s+4/5} \times \frac{16}{10}\frac{-s^2}{s^2-16} \\ &= -\frac{9}{5}\frac{s^2}{(s+4/5)(s-4)} \end{aligned}$$

to perform the pole-zero cancellation.

Then the closed-loop poles are the zeros of:

$$\begin{aligned} 1 + K(s)G(s) &= 1 - \frac{9}{5}\frac{s^2}{(s+4/5)(s-4)} \\ &= 1 - \frac{9}{5}\frac{s^2}{(s^2 - 16/5\, s - 16/5)} \\ &= \frac{s^2 - 16/5\, s - 16/5 - 9/5\, s^2}{(s^2 - 16/5\, s - 16/5)} \\ &= \frac{-\,4/5\, s^2 - 16/5\, s - 16/5}{(s^2 - 16/5\, s - 16/5)} \\ &= -\frac{s^2 + 4s + 4}{5/4\,(s^2 - 16/5\, s - 16/5)} \qquad \text{(+1 point)} \end{aligned}$$

To verify that the closed-loop poles are the same calculate the zeros of $1+K(s)G(s)$ or just plug in $s = -2$:

$$s^2 + 4s + 4 = 4 - 8 + 4 = 0 \qquad \text{(+1 point)}$$

(d) (3 points) Sketch the magnitude and phase of the Bode plot of the loop transfer-function $L(j\omega) = G(j\omega)K(j\omega)$.

Solution:

From part (c):

$$L(s) = -\frac{9}{5}\frac{s^2}{(s+4/5)(s-4)}$$

Normalizing:

$$G(s) = \frac{9}{16}\frac{s^2}{(1 + 5/4\, s)(1 - s/4)} \qquad \text{(+1 point)}$$

In the magnitude plot there will be a line of constant slope $+40$db/decade starting at $\omega \to 0$ and the slope will drop to $+20$db/decade after the pole at $\omega = 4/5$. The straight-line approximations meet at $\omega = 4/5$ at

$$\frac{9}{16}|(j\,4/5)^2| = \frac{9}{16} \times \frac{16}{25} = \frac{9}{25}$$

In dBs:

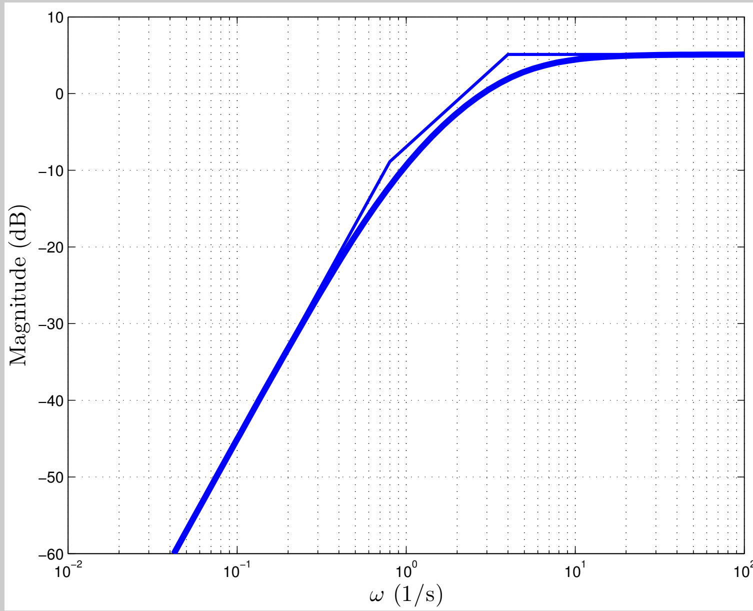
$$20 \log_{10} \frac{9}{25} \approx -9\text{dB}.$$

From $\omega = 4/5$ it continues with a $+20$db/decade slope. The straight-line approximations meet at $\omega = 4$ at

$$-9 + 20\log_{10}\frac{4}{4/5} = -9 + 20\log_{10} 5 \approx -9 + 14 = 5\text{dB}.$$

and then continue with a 0db/decade after $\omega = 4$. (+1/2 point)

The magnitude plot looks like:

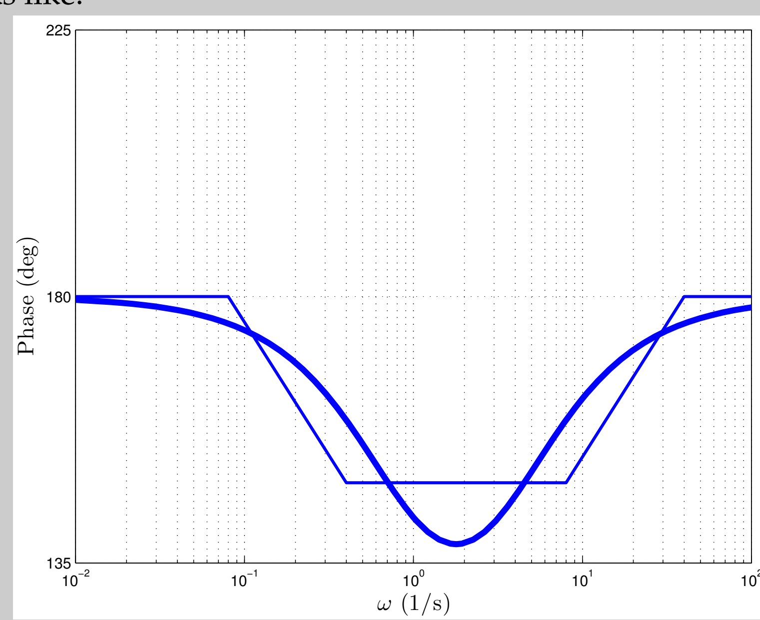


The phase

$$\begin{aligned}\angle L(s) &= \angle\frac{9}{16} + \angle(j\omega)^2 - \angle(1 + j5\omega/4) - \angle(1 - j\omega/4)\\ &= 0 + \pi - \angle(1 + j5\omega/4) - \angle(1 - j\omega/4)\end{aligned}$$

so the plot starts at $180°$ then has a negative slope of $-45°$/decade starting at $\omega = 4/50 \approx 0.08$ due to the pole at $s = 4/5 \approx 0.8$ with negative real part. The slope cancels out at $\omega = 0.4$ due to the pole at $s = 4$ with positive real part. The slope remains at zero until $\omega = 8$ at which point it becomes $+45°$/decade and then back to zero at $\omega = 40$. (+1/2 point)

The plot looks like:



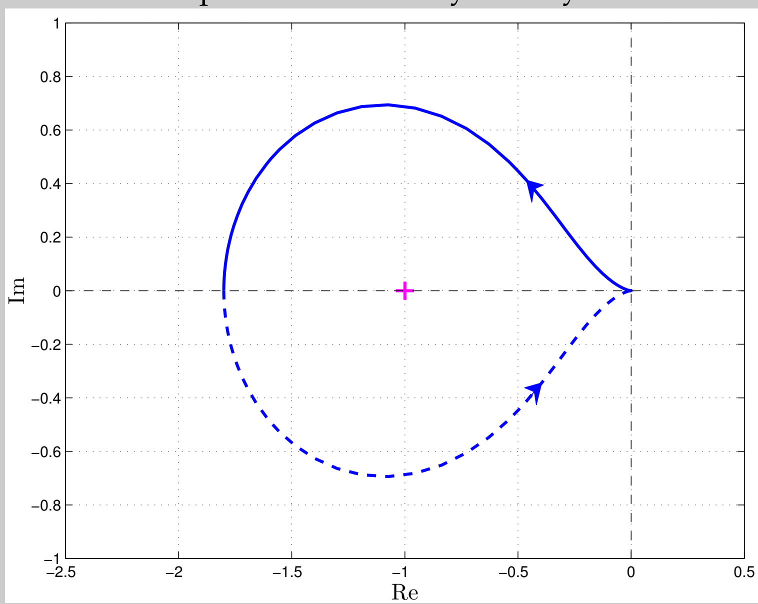
(+1 point)

(e) (2 points (bonus)) Sketch the polar plot of the transfer-function $L(j\omega) = G(j\omega)K(j\omega)$ and explain why the controller $K(s)$ asymptotically stabilizes $G(s)$ using the Nyquist stability criterion.

Solution:

A sketch of the polar plot of $L(s) = K(s)G(s)$ can be obtained from the Bode plot in part (d).

It starts at the origin from where the phase decreases from $180°$ while the magnitude increase until the phase starts increasing again to reach $180°$ at $\omega \to \infty$ at $-9/5 = 1.8$. The rest of the plot comes from symmetry to obtain:



(+1 point)

L has one pole on the right-hand side of the complex plane so that $Z_\gamma = 1$. The polar plot encloses the point -1 once in the clockwise direction so that

$$Z_\Gamma = P_\Gamma + \frac{1}{2\pi}\Delta_\Gamma^{-1/\alpha} \arg L(s) = 1 - 1 = 0$$

and the conclusion is that the closed-loop system is asymptotically stable according to the Nyquist stability criterion.

6. **Reflections.**

(a) (3 points) Why is your instructor looking up in Figure 1? What changes if he balances the broomstick by looking at the middle instead of at the top? Do you expect this to be easier or harder? What feature in your model explains your reasoning? Can he balance the broomstick by looking at his hand instead?

Solution:

The point at which you look at determines q. Because

$$G(s) = \frac{-q\,\sigma\,s^2}{s^2 - g\,\sigma}$$

a smaller q means a smaller gain. (+1 point)

When you're balancing the broomstick the gain will have to be compensated by the controller, that is, you. In this way, you will need to impart twice as much acceleration to provide the same level of stability if you're looking half way into the broomstick. (+1 point)

If you're looking at your hand the gain collapses to zero and it is not possible to control the broomstick any longer. Try that! In this case this is as dramatic as closing your eyes, in which case you will also not be able to control the broomstick. (+1 point)

(b) (2 points) Fix the cross section area and general shape of the broomstick. The mass of the broomstick, m, grows linearly with the length of the broomstick while the moment of inertia, J, grows linearly with the mass and quadratically with the length of the broomstick. Based on this observation, name one reason why it might be easier to control longer broomsticks.

Solution:

The mass and moment of inertia affect the constant σ. Because

$$\sigma = \frac{mr}{J_r} = \frac{mr}{J + mr^2}$$

If ℓ is the length of the broomstick then

$$m \propto \ell, \qquad r \propto \ell, \qquad J \propto m\ell^2 \propto \ell^3.$$

Therefore

$$\sigma = \frac{mr}{J + mr^2} \propto \frac{\ell^2}{\ell^3} \propto \frac{1}{\ell}. \qquad \text{(+1 point)}$$

So larger brooms will have smaller σ, therefore smaller poles. The smaller pole will be easier to control, as the control actions can be slower. (+1 point)

(c) (1 point (bonus)) Continuing part (b), name one reason why it might be harder to control longer broomsticks.

Solution:

If the broom is too large than it may be hard to see the end you're trying to control. In this sense we should expect to be harder to control a bigger broomstick.

(+1 point)