Problems

P2.1. Use Newton's Law to show that the first-order ordinary differential equation:

$$m\,\dot{v} + b\,v = m\,g$$

is a simplified description of the motion of an object of mass m dropping vertically under constant gravitational acceleration, g, and linear air resistance, $-b\,v$.

P2.2. The first-order ordinary differential equation obtained in **P2.1** can be seen as a dynamic system where the output is the vertical velocity, v, and the input is the gravitational force, $m\,g$. Use § 2.3 to calculate the solution to this equation. Consider $m = 1$kg and $b = 10$kg s^{-1} and sketch or use MATLAB to plot the response, $v(t)$, when $v(0) = 0$, $v(0) = 1$m/s, or $v(0) = -1$m/s.

P2.3. Calculate the position, $x(t)$, corresponding to the velocity, $v(t)$, computed in **P2.2**. Using the same data, sketch or use MATLAB to plot the position $x(t)$, when $x(0) = 0$ and $v(0) = 0$, $v(0) = 1$m/s, or $v(0) = -1$m/s.

P2.4. A sky diver weighing 70kg reaches a constant vertical speed of 200km/h during the free-fall phase of the dive and a vertical speed of 20km/h after the parachute is opened. Approximate each phase of the fall by the ordinary differential equation obtained in **P2.1** and estimate the resistance coefficients using the given information. Use $g = 10$m/s^2. What are the time-constants in each phase? At what time and distance from the ground should the parachute be opened if the landing speed is to be less or equal than 29km/h? If a dive starts at a height of 4km with zero vertical velocity at the moment of the jump and the parachute is opened 60s into the dive, how long is the diver airborne?

P2.5. The first-order nonlinear ordinary differential equation:

$$m\,\dot{v} + b\,v^2 = m\,g, \qquad v > 0$$

is a simplified description of the motion of an object of mass m dropping vertically under constant gravitational acceleration, g, and quadratic air resistance, $b\,v^2$. Verify that:

$$v(t) = \frac{1 + \alpha\,e^{\lambda t}}{1 - \alpha\,e^{\lambda t}}\tilde{v}, \quad \tilde{v} = \sqrt{\frac{mg}{b}},$$
$$\alpha = \frac{v_0 - \tilde{v}}{v_0 + \tilde{v}}, \quad \lambda = -\frac{2b\tilde{v}}{m},$$

is a solution to this differential equation.

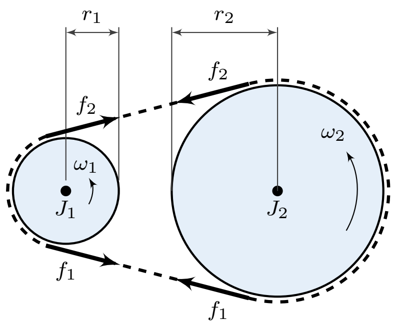


Fig. 2.18: Diagram for **P2.7**

P2.6. Redo problem **P2.4** using the nonlinear model from **P2.5**.

The one-dimensional rotation of a rigid body can be shown to be described by the first-order ordinary differential equation:

$$J\,\dot{\omega} = \tau$$

where ω is the body's angular speed, J is the body's moment of inertia about its center of mass, and τ is the sum of all torques about the center of mass of the body. In constrained rotational systems, e.g. lever, gears, etc, the center of mass can be replaced by the center of rotation.

P2.7. An (inextensible and massless) belt is used to drive a rotating machine without slip as shown in Fig. 2.18. The simplified motion of the inertia J_1 is described by:

$$J_1\,\dot{\omega}_1 = \tau + f_1\,r_1 - f_2\,r_1$$

where τ is the torque applied by the driving motor and f_1 and f_2 are tensions on the belt. The machine is connected to the inertia J_2, which is the sum of the inertia of all machine parts. The motion of the inertia J_2 is described by:

$$J_2\,\dot{\omega}_2 = f_2\,r_2 - f_1\,r_2$$

Show that the motion of the entire system can be described by the differential equation:

$$(J_1 r_1^2 + J_2 r_2^2)\,\dot{\omega}_1 = r_2^2\,\tau.$$

and that $\omega_2 = (\,r_1/r_2\,)\,\omega_1$.

P2.8. Why is $f_1 \neq f_2$ in **P2.7**? Under what conditions are f_1 and f_2 equal?

P2.9. Redo problem **P2.7** in the presence of viscous friction torques, $-b_1\,\omega_1$ and $-b_2\,\omega_2$, on each inertia.

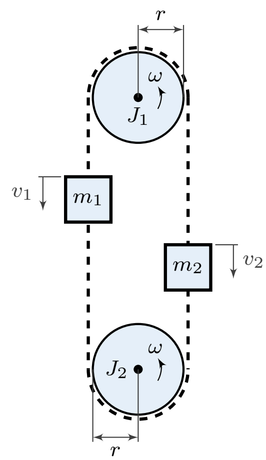
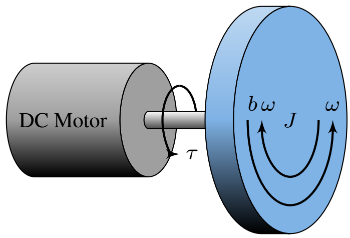
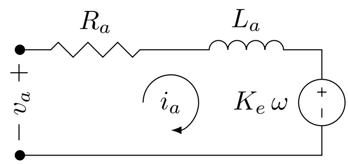


Fig. 2.19: Diagram for **P2.10**



(a) Mechanical



(b) Electrical

Fig. 2.20: Diagrams for **P2.16**

P2.10. A schematic of an elevator is shown in Fig. 2.19. Proceed as in **P2.7** to show that

$$J\,\dot{\omega} + (b_1 + b_2)\,\omega = \tau + g\,r(m_1 - m_2),$$
$$J = J_1 + J_2 + r^2(m_1 + m_2),$$

$v_1 = r\,\omega$, $v_2 = -r\,\omega$, is a simplified description of the motion of the entire elevator system where τ is the torque applied by the driving motor on the inertia J_1, and b_1 and b_2 are viscous friction torque coefficients at the inertias J_1 and J_2. If m_1 is the load to be lifted and m_2 is a counterweight, explain why it is advantageous to have the counterweight as close as possible to the elevator load.

P2.11. Determine a first-order ordinary differential equation based on **P2.10** to describe the elevator as a dynamic system where the output is the vertical velocity of the mass m_1, v_1, and the inputs are the motor torque, τ, and the gravitational torque, $g\,r(m_1 - m_2)$. Use § 2.3 to calculate the solution to this equation. Consider $\tau = 0$, $r = 1$m, $m_1 = m_2 = 1000$kg, $b_1 = b_2 = 120$kg m/s, $J_1 = J_2 = 200$kg m^2, and $g = 10$m/s^2. Sketch or use MATLAB to plot the response, $v_1(t)$, when $v_1(0) = 0$, $v_1(0) = 1$m/s, or $v_1(0) = -1$m/s.

P2.12. Repeat **P2.11** with $m_2 = 800$kg.

P2.13. Calculate the (open-loop) motor torque, τ, for the elevator model in **P2.11** so that the vertical velocity of the mass m_1, v_1, converges to a desired $\bar{v}_1$ as t gets large.

P2.14. The feedback controller:

$$\tau(t) = K(\bar{v}_1 - v_1(t))$$

can be used to control the ascent and descent speed of the mass m_1 in the elevator discussed in **P2.10** and **P2.11**. As in § 2.5, calculate and solve a differential equation that describes the closed-loop response of the elevator. Using data from **P2.11**, select a controller gain, K, with which the time-constant of the elevator is approximately 1s. Calculate the closed-loop steady-state error between the desired vertical velocity, $\bar{v}_1$, and $v_1(t)$.

P2.15. Repeat problem **P2.14** this time with m_2 as in **P2.12**. As in § 2.7, treat the gravitational torque, $g\,r(m_1 - m_2)$, as a disturbance. Calculate the closed-loop steady-state error between the desired vertical velocity, $\bar{v}_1$, and $v_1(t)$.

P2.16. The mechanical motion of the rotor of a DC motor shown schematically in Fig. 2.20a can be described by the differential equation:

$$J\,\dot{\omega} + b\,\omega = \tau,$$

where ω is the rotor angular speed, J is the rotor moment of inertia, b is the coefficient of viscous damping. The rotor torque, τ, is given by:

$$\tau = K_t\,i_a$$

where i_a is the armature current and K_t is the motor torque constant. Neglecting the effects of the armature inductance ($L_a \approx 0$), the current is determined by the circuit in Fig. 2.20b:

$$v_a = R_a\, i_a + K_e\, \omega,$$

where v_a is the armature voltage, R_a is the armature resistance, and K_e is the Back-EMF constant. Combine these equations to show that:

$$J\,\dot{\omega} + \left(b + \frac{K_t K_e}{R_a}\right)\omega = \frac{K_t}{R_a} v_a.$$

This ordinary differential equation can be seen as a dynamic system where the output is the angular velocity, ω, and the input is the armature voltage, v_a. Use § 2.3 to calculate the solution to this equation when v_a is constant. Estimate the parameters of a DC motor that achieves a steady-state angular velocity of 5000RPM when $v_a = 12$V and has a time-constant of 0.1s.

P2.17. What other motor parameters could you estimate in **P2.16** if you knew $R_a = 0.2\Omega$? If necessary, propose experiments to determine the remaining parameters.

P2.18. The feedback controller:

$$v_a(t) = K(\bar{\omega} - \omega(t))$$

can be used to regulate the angular speed of the DC motor, $\omega(t)$, for which a model was developed in **P2.16**. As in § 2.5, calculate and solve a differential equation that describes the closed-loop response of the DC motor. Using data from problem **P2.16**, select a controller gain, K, with which the closed-loop steady-state error between the desired angular speed, $\bar{\omega}$, and $\omega(t)$ is less than 10%. Calculate the resulting closed-loop time-constant and the voltage $v_a(0)$ generated in response to a reference $\bar{\omega} = 4000$RPM assuming zero initial conditions. What is the maximum value of K for which $v_a(0)$ is smaller than 12V?

The next problems have simple examples of heat and fluid flow using ordinary differential equations. Detailed modeling of such phenomena often requires partial differential equations.

P2.19. A substance's temperature, T in K or in °C, flowing in and out of a container kept at the ambient temperature, T_o, with an inflow temperature, T_i, and a heat source, q in W, can be approximated by the differential equation:

$$m\,c\,\dot{T} = q + w\,c\,(T_i - T) + \frac{1}{R}(T_o - T),$$

where m and c are the substance's mass and specific heat, and R is the overall system's thermal resistance. The input and output flow mass rates are assumed to be equal to w in kg/s. This differential equation model can be seen as a dynamic system where the output is the substance's temperature, T, and the inputs are the heat source, q, the flow rate, w, and the temperatures T_0 and T_i. Use § 2.3 to calculate the solution to this equation when q, w, T_o, and T_i are constants.

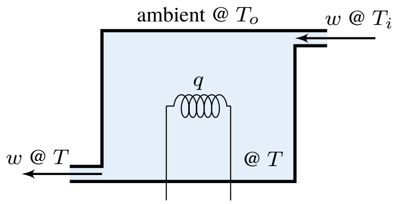


Fig. 2.21: Diagram for **P2.19**

P2.20. Assume that water's density and specific heat are 997.1kg/m^3 and $c = 4186$J/kg K. A 50gal (≈ 0.19m^3) water heater is turned off full with water at 140°F ($\approx$ 60°C). Use the differential equation in **P2.19** to estimate the heater's thermal resistance, R, knowing that after 7 days left at a constant ambient temperature, 77°F ($\approx$ 25°C), without turning it on, $q = 0$, or cycling any water, $w = 0$, the temperature of the water was about 80°F ($\approx$ 27°C). In the same conditions, calculate how much time it takes for the water temperature to reach 80°F ($\approx$ 27°C) with a constant in/out flow of 2gal/h ($\approx 2.1 \times 10^{-6}$m^3/s) at ambient temperature. If the water heater is rated at 40000BTU/h ($\approx$ 12kW) calculate the time it takes to heat up a heater filled up with water from ambient temperature to 140°F ($\approx$ 60°C) first without any in/out flow of water, $w = 0$, then with a constant in/out flow of 2gal/h at ambient temperature.

P2.21. Most residential water heaters have a simple *on/off* type controller: the water heater is turned on at full power when the water temperature, T, falls below a set value, $\underline{T}$, and is turned off when it reaches a second set point, $\overline{T}$. For a heater with the same specifications as in **P2.20** and thermal resistance $R = 0.25$m^{2}K/W, sketch or use MATLAB to plot the temperature of the water during 3 days of a heater filled up with water at ambient temperature, 77°F ($\approx$ 25°C), with an on/off controller set with $\underline{T} = 122$°F ($\approx$ 50°C) and $\overline{T} = 140$°F ($\approx$ 60°C), first without any in/out flow of water, $w = 0$, then with a constant in/out flow of 2gal/h at ambient temperature. Compute the average water tem-

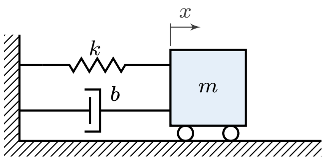


Fig. 2.22: Diagram for **P2.23**

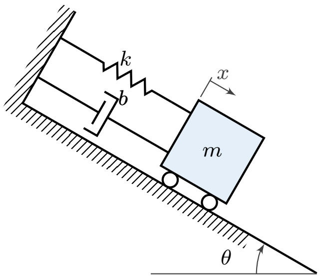


Fig. 2.23: Diagram for **P2.24**

perature over the entire period and over the last 2 days.

P2.22. Repeat **P2.21** with $\underline{T} = 129.2°\text{F}$ ($\approx 54°\text{C}$) and $\overline{T} = 132.8°\text{F}$ ($\approx 56°\text{C}$). What is the impact of the choice of $\underline{T}$ and $\overline{T}$ on the performance of the controller?

Figs. 2.22 through 2.26 show diagrams of mass-spring-damper systems. Assume that there is no friction between the wheels and the floor and, that all springs and dampers are linear: elongating a linear spring with rest length ℓ_0 *by* $\Delta\ell$ *produces an opposing force* $k\Delta\ell$ *(*Hooke's Law*), where* $k > 0$ *is the spring* stiffness*; changing the length of a linear damper at a rate* $\dot{\ell}$ *produces an opposing force* $b\,\dot{\ell}$*, where* b *is the damper's* damping coefficient.

P2.23. Use Newton's Law to show that the second-order ordinary differential equation:

$$m\ddot{x} + b\dot{x} + kx = 0$$

is a simplified description of the motion of the mass-spring-damper system in Fig. 2.22 where the coordinate x is chosen so that $x = 0$ when the length of the spring is equal to its rest length ℓ_0. Why the equations do not depend on the rest length ℓ_0?

P2.24. Use Newton's Law to show that the second-order ordinary differential equation:

$$m\ddot{x} + b\dot{x} + kx = m\,g\sin\theta$$

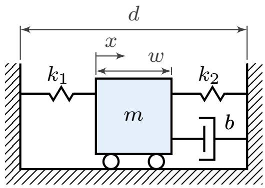


Fig. 2.24: Diagram for **P2.26**

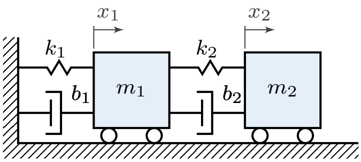


Fig. 2.25: Diagram for **P2.28**

is a simplified description of the motion of the mass-spring-damper system in Fig. 2.23 where g is the gravitational acceleration and the coordinate x is chosen so that $x = 0$ when the length of the spring is equal to its rest length ℓ_0.

P2.25. Rewrite the ordinary differential equation obtained in **P2.24** in terms of:

$$y = x - k^{-1}m\,g\sin\theta$$

Explain the result.

P2.26. Use Newton's Law to show that the second-order ordinary differential equation:

$$m\ddot{x} + b\dot{x} + (k_1 + k_2)x = 0$$

is a simplified description of the motion of the mass-spring-damper system in Fig. 2.24. What is the corresponding choice of x made in this case? Why the equations do not depend on the rest length ℓ_0 nor the dimensions d and w? What is the difference between the cases $d \geq w + 2\ell_0$ and $d \leq w + 2\ell_0$?

P2.27. Can you replace the two springs in **P2.26** by a single spring and still obtain the same ordinary differential equation?

P2.28. Use Newton's Law to show that the second-order ordinary differential equations:

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + k_1x_1 = 0,$$
$$m_2\ddot{x}_2 + b_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0$$

constitute a simplified description of the motion of the mass-spring-damper system in Fig. 2.25

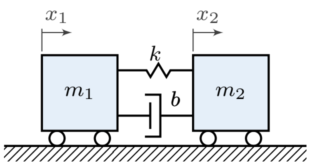


Fig. 2.26: Diagram for **P2.29**

where $x_1 = x_2 = 0$ when the length of both springs is their rest length.

P2.29. Use Newton's Law to show that the second-order ordinary differential equations:

$$m_1\ddot{x}_1 + b(\dot{x}_1 - \dot{x}_2) + k(x_1 - x_2) = 0,$$
$$m_2\ddot{x}_2 + b(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) = 0,$$

constitute a simplified description of the motion of the mass-spring-damper system in Fig. 2.26 where $x_1 = x_2 = 0$ when the length of the spring is its rest length. Show that it is possible to *decouple* the equations if you write them in the coordinates

$$y_1 = x_1 + x_2, \qquad y_2 = x_1 - x_2.$$

Explain why using what you know from physics. What happens when $m_1 = m_2$?