Problems

P3.1. Compute the Laplace transform of the following signals $f(t)$ defined for $t \geq 0$:

a) $1 - e^{-t}$;

b) $e^{-t} - e^{-2t}$;

c) $\sinh(t) - \cosh(2t)$;

d) $\sin(t) - \cos(t)$;

e) $t + t\,e^{-t}$;

f) $\sin(t) - e^{-t}\cos(2t)$;

g) $e^{-t}\cos(t + \pi/4)$;

h) $1(t) + 1(t-1)$;

i) $e^{-t} + 1(t-1)\,e^{t-1}$;

j) $\delta(t) + t\,e^{-t}\,\sin(t)$;

P3.2. Compute the inverse Laplace transform of the following complex valued functions $F(s)$:

a) $\dfrac{1}{s(s+1)}$;

b) $\dfrac{s-1}{s+1}$;

c) $\dfrac{s-1}{s(s+1)}$;

d) $\dfrac{s}{s^2+2s+1}$;

e) $\dfrac{1}{s^2-1}$;

f) $\dfrac{1}{s^2(s+2)^2}$;

g) $\dfrac{1}{(s+1)^2+1}$;

h) $\dfrac{1-e^{-s}}{s}$;

i) $\dfrac{s+1-e^{-s}}{s(s+1)}$;

j) $\dfrac{1}{s+1} - \dfrac{s}{(s+1)^2} + \dfrac{s^2}{(s+1)^3}$;

P3.3. Prove that when $f(t)$, $t \geq 0$, is such that $f(t) = 0$, for all $t \geq T$ then its Laplace transform is analytic in $\mathbb{C}$ (entire) or its singularities are *removable*. Verify this by showing that the following complex valued functions:

a) $1 - e^{-s} + e^{-2s}$;

b) $\dfrac{1-e^{-s}}{s}$;

c) $\dfrac{1-2e^{-s}+e^{-2s}}{s}$;

d) $\dfrac{1-2e^{-s}+e^{-2s}}{s^2}$;

e) $\dfrac{1-2e^{-s}+2e^{-3s}-e^{-4s}}{s^2}$;

f) $\dfrac{1-2e^{-s}+2e^{-3s}-e^{-4s}}{s^3}$;

g) $\dfrac{e^{-s-1}(e^s-e)}{s-1}$;

have only removable singularities. Compute the inverse Laplace transform, $f(t) = \mathcal{L}^{-1}\{F(s)\}$, and sketch the plot of $f(t)$, $t \geq 0$.
Hint: Roughly speaking, $F(s)$ is singular at a removable singular point s_0 if $\lim_{s\to s_0} F(s_0)$ exists and is bounded and $F'(s_0)$ and all higher order derivative exists, e.g. $s^{-1}\sin(s)$ at 0.

P3.4. The complex valued functions $F(s)$:

a) $\dfrac{e^{-s}}{s}$;

b) $\dfrac{1-e^{-2\pi s}}{s^2+1}$;

c) $\dfrac{s(1-e^{-2\pi s})}{s^2+1}$;

are Laplace transforms obtained from functions $f(t)$, $t \geq 0$. Assume that $f(0^-) = 0$ and use the formal derivative property and the inverse Laplace transform to calculate the time-derivative function, $\dot{f}(t)$, $t \geq 0$, without evaluating $f(t)$. Locate the discontinuities of the original function $f(t)$.

P3.5. Show that

$$\mathcal{L}\left\{\frac{\sin(at)}{t}\right\} = \tan^{-1}\left(\frac{a}{s}\right).$$

Hint: Use $\int_0^a \cos(at)\,da = \sin(at)/t$ and the linearity property.

P3.6. Use **P3.5**, the (t-domain) integration property and the final value property to show that

$$\lim_{t\to\infty}\int_{0-}^{t}\frac{\sin(at)}{t}\,dt = \mathrm{sign}(a)\frac{\pi}{2}.$$

P3.7. Use **P3.6** and (3.7) to show that

$$\int_{-\infty}^{\infty}\delta(t)\,dt = 1.$$

P3.8. Use the convolution property to show that

$$\int_{0-}^{t} f(\tau)\,\delta(t-\tau)\,d\tau = f(t).$$

P3.9. Let C be the unit circle traversed in the counter-clockwise direction. Use Theorem 3.1, Cauchy's Residue Theorem, to evaluate the contour integral:

$$\int_C f(s)\, ds$$

for the following complex valued functions:

a) $1 - e^{-s}$;

b) $\dfrac{1 + e^{-s}}{s}$;

c) $\dfrac{1}{s}$;

d) $\dfrac{1}{s^2}$;

e) $\dfrac{1}{s(s + 1/2)}$;

f) $\dfrac{s}{(s + 1/2)(s + 2)}$;

P3.10. Compute the expansion in partial fractions of the following rational functions:

a) $\dfrac{1}{(s+1)(s+2)}$;

b) $\dfrac{s-1}{(s+1)(s+2)}$;

c) $\dfrac{s(s-1)}{(s+1)(s+2)}$;

d) $\dfrac{1}{s(s+1)(s+2)}$;

e) $\dfrac{1}{s^2(s+1)}$;

f) $\dfrac{1}{s^2 + 2s + 1}$;

g) $\dfrac{1}{s^2 + 2s + 2}$;

h) $\dfrac{s+1}{(s^2 + 2s + 2)^2}$;

P3.11. Let $u(t)$, $t \geq 0$, be such that $u(t) = 0$ for all $t \geq T$ and $U(s)$ be its Laplace transform. Show that

$$y(t) = \mathcal{L}^{-1}\{U(s)\}, \quad Y(s) = \frac{U(s)}{1 - e^{-sT}}$$

is a periodic function with period T. Use this fact to calculate and sketch the plot of the inverse Laplace transform of

$$Y(t) = \frac{1 - e^{-s\,T/2}}{s(1 - e^{-sT})}.$$

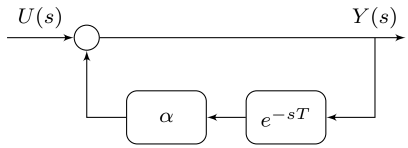

Fig. 3.4: Block diagram for **P3.12**

Repeat for

$$Y(t) = \frac{1 - e^{-s\,3T/2}}{s(1 - e^{-sT})}.$$

Explain what happens when $u(t) \neq 0$, $t \geq T$? *Hint: Recall that* $(1-x)^{-1} = 1 + x + x^2 + \cdots$, *for all* $|x| < 1$.

P3.12. Calculate $Y(s)$ in the block-diagram in Fig. 3.4. What happens when $\alpha = 1$? What about $\alpha < 1$ and $\alpha > 1$? Compare with **P3.11**.

P3.13. Use the Laplace transform to solve the following linear ordinary differential equations:

a) $\dot{y} + y = 0, \quad y(0) = 1;$

b) $\dot{y} + y = 1, \quad y(0) = 0;$

c) $\dot{y} - y = 0, \quad y(0) = 1;$

d) $\dot{y} - y = 1, \quad y(0) = 0;$

e) $\ddot{y} + \dot{y} = 0, \quad y(0) = 1, \dot{y}(0) = -1;$

f) $\ddot{y} + \dot{y} = e^{-t}, \quad y(0) = \dot{y}(0) = 0;$

g) $\ddot{y} + y = 0, \quad y(0) = 1, \dot{y}(0) = -1;$

h) $\ddot{y} + y = \cos(t), \quad y(0) = \dot{y}(0) = 0;$

i) $\ddot{y} + 2\,\dot{y} + y = 0, \quad y(0) = 1, \dot{y}(0) = 0;$

j) $\ddot{y} + 2\,\dot{y} + y = \sin(t), \quad y(0) = \dot{y}(0) = 0;$

k) $\ddot{y} + 2\,\dot{y} + 2\,y = 0, \quad y(0) = 1, \dot{y}(0) = 0;$

l) $\ddot{y} + 2\,\dot{y} + 2\,y = \cos(t), y(0) = \dot{y}(0) = 0;$

P3.14. A *sample-and-hold* system produces the output

$$y(t) = u(kT), \qquad kT \leq t < (k+1)\,T, \quad k \in \mathbb{N},$$

in response to a continuous input, $u(t)$. Show that the sample-and-hold system is a linear system. Show that it is not time-invariant. Why the name *sample-and-hold*? Can you think of an application for such system?

P3.15. Show that the response of sample-and-hold system, **P3.14**, to an impulse $\delta(t-\tau)$ is

$$g(t,\tau)=\delta(k\,T-\tau),\qquad k\,T\le t<(k+1)\,T,\quad k\in\mathbb{N}.$$

and that

$$y(t)=\int_0^t g(t,\tau)\,u(\tau)\,d\tau.$$

*Hint: Use **P3.7**.*

In the following questions, when asked to compute the response to a given input, assume zero initial conditions unless otherwise noted.

P3.16. A linear time-invariant system has as impulse response

$$g(t)=e^{-t},\qquad t\ge 0.$$

Compute the system's transfer-function. What is the order of the system? Is the system asymptotically stable? Calculate the response to a constant input $u(t)=\tilde{u}$, $t\ge 0$. Identify the transient and steady-state part of the response.

P3.17. Repeat **P3.16** for the following impulse responses, $g(t)$, $t\ge 0$:

a) $e^{-2t}-e^{-t}$;

b) $e^{t}-e^{-t}$;

c) $-te^{-2t}$;

d) $e^{-t}\cos(t)$;

e) $e^{-t}(\cos(t)-\sin(t))$;

f) $\cos(t+\pi/6)$;

g) $\delta(t)+t\,e^{-t}\sin(t)$;

h) $1(t)-1(t-1)$;

i) $t-2(t-1)1(t-1)+(t-2)1(t-2)$;

P3.18. You have shown in **P2.1** that the ordinary differential equation

$$m\,\dot{v}+b\,v=m\,g$$

is a simplified description of the motion of an object of mass m dropping vertically under constant gravitational acceleration, g, and linear air resistance, $-b\,v$. Solve this differential equation using the Laplace transform. Treating the gravitational force, $u=mg$, as an input, calculate the transfer-function from the input, u, to the velocity, v, then calculate the transfer-function from the input, u, to the object's position, $x=\int_0^t v(\tau)\,d\tau$. Assume that all constants are positive. Are these transfer-functions asymptotically stable?

P3.19. You have shown in **P2.7** that the ordinary differential equation

$$(J_1r_1^2+J_2r_2^2)\,\dot{\omega}_1=r_2^2\,\tau,\qquad \omega_2=(\,r_1/r_2\,)\,\omega_1$$

is a simplified description of the motion of a rotating machine driven by a belt without slip as in Fig. 2.18, where ω_1 is the angular velocity of the driving shaft and ω_2 is the machine's angular velocity. Calculate the transfer-function from the input torque, τ, to the machine's angular velocity, ω_2, then calculate the transfer-function from the torque, τ, to the machine's angular position, $\theta_2=\int_0^t\omega_2(\tau)\,d\tau$. Assume that all constants are positive. Are these transfer-functions asymptotically stable?

P3.20. Consider the simplified model of the rotating machine described in **P3.19**. Calculate the machine's angular position, θ_2, and angular velocity, ω_2, obtained in response to a constant torque input, $\tau(t)=\tilde{\tau}$, $t\ge 0$. Identify the transient and the steady-state component of the responses. Repeat the question for an input $\tau(t)=\tilde{\tau}\cos(\omega t)$, $t\ge 0$.

P3.21. Recalculate the steady-state responses to the inputs in **P3.20** using the system's frequency response. Compare your answers.

P3.22. You have shown in **P2.7** that the ordinary differential equation

$$J\,\dot{\omega}+(b_1+b_2)\,\omega=\tau+g\,r(m_1-m_2),$$
$$J=J_1+J_2+r^2(m_1+m_2),$$
$$v_1=r\,\omega,$$

is a simplified description of the motion of the elevator in Fig. 2.19, where ω is the angular velocity of the driving shaft and v_1 is the elevator's load linear velocity. Treating the gravitational torque, $w=g\,r(m_1-m_2)$, as an input, calculate the transfer-function, G_w, from the gravitational torque, w, to the elevator's load linear velocity, v_1, assuming that the motor torque, τ, is zero. Calculate the transfer-function, G_τ, from the motor torque, τ, to the the elevator's load linear velocity, v_1, assuming that the gravitational torque, w, is zero. Assume that all constants are positive. Are these transfer-functions asymptotically stable? Assume zero initial conditions and verify that:

$$V_1(s)=G_\tau(s)\,\tau(s)+G_w(s)\,W(s).$$

Explain the results.

P3.23. Consider the simplified model of the elevator described in **P3.22**. Calculate the elevator's linear velocity, v_1, obtained in response to a constant torque input, $\tau(t) = \tilde{\tau}$, $t \geq 0$. Identify the transient and the steady-state component of the response. Repeat the question for an input $\tau(t) = \tilde{\tau} \cos(\omega t)$, $t \geq 0$.

P3.24. Recalculate the steady-state responses to the inputs in **P3.23** using the system's frequency response. Compare your answers.

P3.25. You have shown in **P2.16** that the ordinary differential equation

$$J \, \dot{\omega} + \left(b + \frac{K_t K_e}{R_a} \right) \omega = \frac{K_t}{R_a} v_a$$

is a simplified description of the motion of the rotor of the DC motor in Fig. 2.20, where ω is the rotor angular velocity. Calculate the transfer-function from the armature voltage, v_a, to the the rotor's angular velocity, ω. Assume that all constants are positive. Is this transfer-function asymptotically stable?

P3.26. Consider the simplified model of the DC motor described in **P3.25**. Calculate the motor's angular velocity, ω, obtained in response to a constant armature voltage input, $v_a(t) = \tilde{v}_a$, $t \geq 0$. Identify the transient and the steady-state component of the response. Repeat the question for $v_a(t) = \tilde{v}_a \cos(\omega t)$, $t \geq 0$.

P3.27. Recalculate the steady-state responses to the inputs in **P3.26** using the system's frequency response. Compare your answers.