MAE 143B - Homework 3

3.2

b)

Rewrite

$$F(s) = \frac{s-1}{s+1} = 1 - \frac{2}{s+1},$$

such that

$$\mathcal{L}^{-1}\left\{F(s)\right\} = \mathcal{L}^{-1}\left\{1\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = \delta(t) - 2e^{-t}$$

for all $t \geq 0$.

h)

Using the time shift rule in Tab. 3.2, we have

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} = 1(t) - 1(t-1).$$

3.9 - f)

By Cauchy's Residue Theorem, we know

$$\int_C f(s)ds = 2\pi j \sum_{k=1}^{n} \operatorname*{Res}_{s=p_k}\left\{f(s)\right\}. \tag{1}$$

Now notice that the only singular point inside the unit unit circle is $p_1 = -1/2$. The singular point $p_2 = -2$ lies outside the unit circle. Hence, (1) reduces to

$$\int_C f(s)ds = 2\pi j \operatorname*{Res}_{s=-1/2}\left\{f(s)\right\}, \tag{2}$$

where the residue at p_1 is

$$\operatorname*{Res}_{s=-1/2}\left\{f(s)\right\} = \lim_{s\to-1/2}\left((s+1/2)\cdot f(s)\right) = \lim_{s\to-1/2}\frac{s}{s+2} = -1/3.$$

Plugging into (2) yields the result

$$\int_C f(s)ds = -\frac{2}{3}\pi j.$$

3.10 - e)

The function $F(s)$ has a simple pole at $p_1 = -1$ and a double pole at $p_2 = 0$. Hence, Cauchy's Residue Theorem tells us that

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = \sum_{k=1}^{2} \operatorname*{Res}_{s=p_k} \left\{F(s)e^{st}\right\}.$$

Taking Laplace transforms, we get an expression in the desired form:

$$F(s) = \mathcal{L}\left\{\sum_{k=1}^{2} \operatorname*{Res}_{s=p_k} \left\{F(s)e^{st}\right\}\right\}. \tag{3}$$

The residue at the simple pole p_1 is

$$\operatorname*{Res}_{s=-1} \left\{F(s)e^{st}\right\} = \lim_{s\to-1}(s+1)F(s)e^{st} = \lim_{s\to-1}\frac{1}{s^2}e^{st} = e^{-t}.$$

To compute the residue corresponding to the double p_2, we have to proceed as outlined on page 52 of your notes. That is, we define $G(s) := (s-p_2)^2F(s)e^{st}$ and compute

$$\frac{G^{(1)}(p_2)}{1!} = G'(p_2) = \lim_{s\to0}\frac{d}{ds}\frac{1}{s+1}e^{st} = \lim_{s\to0}\left(\frac{t}{s+1}e^{st} - \frac{1}{(s+1)^2}e^{st}\right) = t-1.$$

Plugging the residues into (3), we get

$$F(s) = \mathcal{L}\left\{e^{-t}+t-1\right\} = \frac{1}{s+1} + \frac{1}{s^2} - \frac{1}{s}.$$

3.13 - i)

Taking the Laplace transform, we get

$$\left(s^2Y(s) - sy(0) - \dot{y}(0)\right) + 2\left(sY(s) - y(0)\right) + Y(s) = \left(s^2+2s+1\right)Y(s) - s - 2 = 0,$$

which yields

$$Y(s) = \frac{s+2}{s^2+2s+1} = \frac{s}{(s+1)^2} + \frac{2}{(s+1)^2}.$$

Taking inverse Laplace transforms now gives the result

$$y(t) = \operatorname*{Res}_{s=-1}\left\{Y(s)e^{st}\right\} = \lim_{s\to-1}\frac{d}{ds}(s+2)e^{st} = \lim_{s\to-1}\left(e^{st} + (s+2)te^{st}\right) = e^{-t}(t+1)$$

for $t \geq 0$. As desired, the initial conditions of $y(t)$ are

$$y(0) = e^0 = 1 \quad \text{and} \quad \dot{y}(0) = -e^0 + e^0 = 0.$$

3.22

To avoid confusion between $w(t)$ and $\omega(t)$, let $z(t)$ instead of $w(t)$ denote the gravitational torque. That is, we define $z(t) := gr(m_1 - m_2)$. Using $z(t)$, $\omega(t) = v_1(t)/r$ and $\tau = 0$, our differential equation now reads

$$\dot{v_1}(t) + \frac{b_1+b_2}{J}v_1(t) = \frac{r}{J}z(t).$$

To determine the transfer function G_z from gravitational torque $z(t)$ to linear velocity $v_1(t)$, we assume zero initial conditions, take the Laplace transform

$$sV_1(s) + \frac{b_1 + b_2}{J} V_1(s) = \frac{r}{J} Z(s)$$

and solve for $V_1(s)/Z(s)$:

$$G_z(s) = \frac{V_1(s)}{Z(s)} = \frac{r}{J} \left(s + \frac{b_1 + b_2}{J} \right)^{-1} = \frac{r}{sJ + b_1 + b_2}.$$

Following the same steps, we get the second transfer function:

$$G_\tau(s) = \frac{V_1(s)}{\tau(s)} = G_z(s) = \frac{r}{sJ + b_1 + b_2}.$$

Both transfer functions have a simple pole at $s_1 = -(b_1 + b_2)/J$. If all constants are positive, we have $\Re\{s_1\} = s_1 < 0$, implying asymptotically stable behavior for both transfer functions. Going back to our initial differential equation and taking Laplace transform for zero intial conditions results in

$$sV_1(s) + \frac{b_1 + b_2}{J} V_1(s) = \frac{r}{J} Z(s) + \frac{r}{J} \tau(s),$$

verifying

$$\begin{aligned} V_1(s) &= \frac{r}{J} \left(s + \frac{b_1 + b_2}{J} \right)^{-1} (Z(s) + \tau(s)) \\ &= \frac{r}{sJ + b_1 + b_2} (Z(s) + \tau(s)) \\ &= G_\tau(s)\tau(s) + G_z(s)Z(s). \end{aligned} \tag{4}$$

The final statement displays how each input influences the output separately. This general property of linear systems is also called the superposition principle. In fact, by $G_\tau(s) = G_z(s)$, both inputs have exactly the same effects on the output signal in this particular example.

3.23

In this problem, we are concerned with outputs resulting from two particular torque input signals. By linearity (see previous problem), we can neglect the gravitational torque, that is $z(t) = 0$ for all $t \geq 0$. First let $\tau(t) = \tilde{\tau}$ for $t \geq 0$. Taking the Laplace transform gives $\tau(s) = \tilde{\tau}/s$, such that (4) for the given torque reads

$$V_1(s) = G_\tau(s)\tau(s) = \frac{1}{s + (b_1 + b_2)/J} \cdot \frac{1}{s} \cdot \frac{r\tilde{\tau}}{J}.$$

Taking the inverse Laplace transform then yields

$$v_1(t) = \operatorname*{Res}_{s=-(b_1+b_2)/J} \left\{ V_1(s)e^{st} \right\} + \operatorname*{Res}_{s=0} \left\{ V_1(s)e^{st} \right\} = \frac{r\tilde{\tau}}{b_1 + b_2} \left(1 - e^{-\frac{b_1+b_2}{J}t} \right)$$

for $t \geq 0$. By boundedness of $v_1(t)$, the steady-state component of the response can be obtained by analyzing $v_1(t)$ for large values of t:

$$v_{1,ss}(t) = \lim_{t\to\infty} v_1(t) = \frac{r\tilde{\tau}}{b_1 + b_2}$$

for $t \geq 0$. The transient response is the remainder of $v_1(t)$, such that

$$v_{1,t}(t) = v_1(t) - v_{1,ss}(t) = -\frac{r\tilde{\tau}}{b_1 + b_2} e^{-\frac{b_1+b_2}{J}t}$$

for $t \geq 0$.

Now consider the second input signal, $\tau(t) = \tilde{\tau}\cos(\omega t)$ for $t \geq 0$, having Laplace transform

$$\tau(s) = \frac{s}{s^2 + \omega^2}\tilde{\tau},$$

such that

$$V_1(s) = G_\tau(s)\tau(s) = \frac{s}{(s^2 + \omega^2)(s + (b_1 + b_2)/J)} \cdot \frac{r\tilde{\tau}}{J}.$$

Now notice that $V_1(s)$ has simple poles at $p_1 = -(b_1+b_2)/J$ and $p_{2,3} = \pm j\omega$ with respective residues

$$\begin{aligned}
\operatorname*{Res}_{s=-(b_1+b_2)/J} \left\{V_1(s)e^{st}\right\} &= \frac{-(b_1 + b_2)r\tilde{\tau}}{(b_1 + b_2)^2 + J^2\omega^2} e^{-\frac{b_1+b_2}{J}t}; \\
\operatorname*{Res}_{s=j\omega} \left\{V_1(s)e^{st}\right\} &= \frac{r\tilde{\tau}}{2(b_1 + b_2 + j\omega J)} e^{j\omega t} = \frac{r\tilde{\tau}}{2} \cdot \frac{b_1 + b_2 - j\omega J}{(b_1 + b_2)^2 + J^2\omega^2} e^{j\omega t}; \\
\operatorname*{Res}_{s=-j\omega} \left\{V_1(s)e^{st}\right\} &= \frac{r\tilde{\tau}}{2(b_1 + b_2 - j\omega J)} e^{-j\omega t} = \frac{r\tilde{\tau}}{2} \cdot \frac{b_1 + b_2 + j\omega J}{(b_1 + b_2)^2 + J^2\omega^2} e^{-j\omega t}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
v_1(t) &= \sum_{k=1}^{3} \operatorname*{Res}_{s=p_k} \left\{V_1(s)e^{st}\right\} \\
&= \frac{r\tilde{\tau}}{(b_1 + b_2)^2 + J^2\omega^2} \left((b_1 + b_2) \left(\frac{1}{2}(e^{j\omega t} + e^{-j\omega t}) - e^{-\frac{b_1+b_2}{J}t} \right) - \frac{J\omega j}{2}(e^{j\omega t} - e^{-j\omega t}) \right) \\
&= \frac{r\tilde{\tau}}{(b_1 + b_2)^2 + J^2\omega^2} \left((b_1 + b_2) \left(\cos(\omega t) - e^{-\frac{b_1+b_2}{J}t} \right) + \frac{J\omega}{2j}(e^{j\omega t} - e^{-j\omega t}) \right) \\
&= \frac{r\tilde{\tau}}{(b_1 + b_2)^2 + J^2\omega^2} \left((b_1 + b_2) \left(\cos(\omega t) - e^{-\frac{b_1+b_2}{J}t} \right) + J\omega \sin(\omega t) \right)
\end{aligned}$$

for $t \geq 0$. In this case, the steady-state component of our output is not constant but oscillating instead. However, we can exploit boundedness of $v_1(t)$ once more to find the steady-state response as the component that does not decay for large t:

$$v_{1,ss}(t) = \frac{r\tilde{\tau}}{(b_1 + b_2)^2 + J^2\omega^2} \left((b_1 + b_2)\cos(\omega t) + J\omega \sin(\omega t) \right)$$

for $t \geq 0$. As above, we then get

$$v_{1,t}(t) = v_1(t) - v_{1,ss}(t) = -\frac{(b_1 + b_2)r\tilde{\tau}}{(b_1 + b_2)^2 + J^2\omega^2} e^{-\frac{b_1+b_2}{J}t}$$

for $t \geq 0$.