MAE 143B, Spring 2014, HW # 4

P3.16: According to the problem statement, we can consider the following block diagram:

$$U(s) \longrightarrow \boxed{F(s)} \longrightarrow Y(s)$$

where $U(s)$ is the laplace transform of $u(t)$, and $Y(s)$ is the laplace transform of $y(t)$, and $F(s)$ is the transfer f'n describing the associated system.

In the first part of the problem, the impulse response is given as $g(t) = e^{-t}$, $t \geq 0$. We then recall that for an impulse, $u(t) = \delta(t)$, the laplace transform is given as $U(s) = \mathcal{L}\{\delta(t)\} = 1$, therefore, the following holds:

$$g(t) = e^{-t} \Rightarrow \mathcal{L}\{g(t)\} = Y(s) = \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

In addition, given the above block diagram:

$$\left.\begin{array}{l} Y(s) = U(s)\,F(s) \\ U(s) = 1 \end{array}\right\} \Rightarrow F(s) = Y(s) = \frac{1}{s+1} \Rightarrow \boxed{F(s) = \frac{1}{s+1}} \quad (*)$$

Thus, $(*)$ is the transfer f'n of the system. The order of the system is then 1 provided its # of poles. This system is also asymptotically stable, provided its only pole is $p_1 = -1 < 0$.

In order to compute the response to a constant input, $u(t) = \tilde{u}$, $t \geq 0$, we recall that:

$$u(t) = \tilde{u} \Rightarrow U(s) = \mathcal{L}\{u(t)\} = \frac{\tilde{u}}{s} \Rightarrow Y(s) = F(s)\,U(s) \xrightarrow[(*)]{\text{by}} \underline{Y(s) = \left(\frac{1}{s+1}\right)\left(\frac{\tilde{u}}{s}\right)}$$

We then split $Y(s)$ into $Y_-(s)$ and $Y_0(s)$:

$$Y(s) = \frac{\tilde{u}}{s(s+1)} = \tilde{u}\left(\frac{1}{s} - \frac{1}{s+1}\right) \Rightarrow Y(s) = \underbrace{\left(\frac{\tilde{u}}{s}\right)}_{Y_0(s)} + \underbrace{\left(-\frac{\tilde{u}}{s+1}\right)}_{Y_-(s)} \quad (**)$$

Therefore, we get:

$$y_{tr} = \mathcal{L}^{-1}\{Y_-(s)\} = \mathcal{L}^{-1}\left\{\frac{-\tilde{u}}{s+1}\right\} \Rightarrow \boxed{y_{tr}(t) = -\tilde{u}\,e^{-t}}$$: transient part of the response

$$y_{ss} = \mathcal{L}^{-1}\{Y_0(s)\} = \mathcal{L}^{-1}\left\{\frac{\tilde{u}}{s}\right\} \Rightarrow \boxed{y_{ss} = \tilde{u}\,1(t)}$$: steady-state part of the response

$$\Rightarrow y(t) = y_{tr}(t) + y_{ss}(t) \Rightarrow \boxed{y(t) = \tilde{u}\,(1(t) - e^{-t})}$$: overall response

P3.17-g:

The steps to solve this problem are similar to P3.16 stated on the previous page; and thus, we state the major steps here.

$$U(s) \longrightarrow \boxed{F(s)} \longrightarrow Y(s)$$

$$\left.\begin{aligned} &u(t)=\delta(t) \Rightarrow U(s)=1 \\ &y(t)=\delta(t)+te^{-t}\sin(t) \Rightarrow Y(s)=\mathcal{L}\{\delta(t)+te^{-t}\sin(t)\}=\mathcal{L}\{\delta(t)\}+\mathcal{L}\{te^{-t}\sin(t)\}\end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} &F(s)=\frac{Y(s)}{U(s)}=\mathcal{L}\{\delta(t)\}+\mathcal{L}\{te^{-t}\sin(t)\}=1+H(s+1) \\ &\text{where } H(s)=\mathcal{L}\{t\sin(t)\}=\frac{2s}{(s^2+1)^2}\end{aligned}\right\} \Rightarrow F(s)=1+\frac{2(s+1)}{((s+1)^2+1)^2}$$

$$\Rightarrow \boxed{F(s)=\frac{((s+1)^2+1)^2+2(s+1)}{((s+1)^2+1)^2}} \quad (*)$$

- The order of the system is 4, provided the order of the denominator is 4.
- The system is asymptotically stable, provided it has all its poles on the LHP, i.e., w/ negative real part.
- Let $u(t)=\tilde{u}$, $t\geq 0 \Rightarrow U(s)=\frac{\tilde{u}}{s} \xrightarrow{(*)} Y(s)=\frac{((s+1)^2+1)^2+2(s+1)}{((s+1)^2+1)^2}\times\frac{\tilde{u}}{s}$

$$\Rightarrow Y(s)=\frac{((s+1)^2+1)^2+2(s+1)}{s((s+1)^2+1)^2}\tilde{u} \Rightarrow \boxed{y(t)=\mathcal{L}^{-1}\{Y(s)\}=\sum_{k=1}^{3}\operatorname*{Res}_{s=p_k}\left(Y(s)e^{st}\right)} \quad (**)$$

w/ $\begin{cases} p_1=0,\ m_1=1, \\ p_2=-1+j,\ m_2=2, \\ p_3=-1-j,\ m_3=2, \end{cases}$ the next step that is computing residues can be performed now.

$$Y(s)e^{st}=\frac{((s+1)^2+1)^2+2(s+1)}{((s+1)^2+1)^2}\times\frac{\tilde{u}}{s}\times e^{st}=\left(\frac{((s+1)^2+1)^2+2(s+1)}{((s+1)^2+1)^2}\times\tilde{u}\times e^{st}\right)\frac{1}{s}$$

$$\Rightarrow \boxed{\operatorname*{Res}_{s=0}\{Y(s)e^{st}\}=\left(\frac{((s+1)^2+1)^2+2(s+1)}{((s+1)^2+1)^2}\times\tilde{u}\times e^{st}\right)\Bigg|_{s=0}=\frac{4+2}{4}\times\tilde{u}\times 1=\frac{3}{2}\tilde{u}} \quad (\bigstar)$$

$$Y(s)\,e^{st} = \frac{((s+1)^2+1)^2 + 2(s+1)}{((s+1)^2+1)^2} \times \frac{\tilde{u}}{s} \times e^{st} = \left(\frac{((s+1)^2+1)^2 + 2(s+1)}{s} \times \tilde{u} \times e^{st} \right) \frac{1}{((s+1)^2+1)^2}$$

$$\Rightarrow \boxed{\operatorname*{Res}_{s=-1+j}\left\{ Y(s)\,e^{st} \right\} = \frac{d}{ds}\left(\frac{((s+1)^2+1)^2 + 2(s+1)}{s} \times \tilde{u} \times e^{st} \right)\Bigg|_{-1+j} = (1-2j)\,\tilde{u}\,e^{(-1+j)t}} \quad (**)$$

$$\Rightarrow \boxed{\operatorname*{Res}_{s=-1-j}\left\{ Y(s)\,e^{st} \right\} = \frac{d}{ds}\left(\frac{((s+1)^2+1)^2 + 2(s+1)}{s} \times \tilde{u} \times e^{st} \right)\Bigg|_{-1-j} = (1+2j)\,\tilde{u}\,e^{(-1-j)t}} \quad (***)$$

$y(t)$ can be obtained by plugging (*), (**), and (***) back in (**):

$$y(t) = \frac{3}{2}\tilde{u} + (1-2j)\,\tilde{u}\left(e^{-t}(\cos t + j\sin t)\right) + (1+2j) \times \tilde{u}\left(e^{-t}(\cos t - j\sin t)\right)$$

$$\Rightarrow y(t) = \frac{3}{2}\tilde{u} + \left[1-2j+1+2j\right]\tilde{u}\,e^{-t}\cos t + \left[j+2-j+2\right]\tilde{u}\,e^{-t}\sin t$$

$$\Rightarrow \boxed{y(t) = \frac{3}{2}\tilde{u} + 2\tilde{u}\,e^{-t}\cos t + 4\tilde{u}\,e^{-t}\sin t} \quad (+)$$

Given (+):

$$\begin{cases} \boxed{y_{tr}(t) = 2\tilde{u}\,e^{-t}(2\sin t + \cos t)} \\ \boxed{y_{ss}(t) = 3/2\,\tilde{u}} \end{cases}$$

∎

P3.24:

We refer to P 3.23 and P 3.22 where we have derived the following:

$$v_1(s) = G_\tau(s)\,\tau(s) + G_w(s)\,W(s),$$

w/

$$G_\tau(s) = \frac{r}{sJ + b_1 + b_2}, \text{ and } G_w(s) = \frac{r}{sJ + b_1 + b_2}.$$

In this problem we shall consider $w(t) \equiv 0$ and thus we consider only the $\tau(s)$ part:

$$\boxed{V_1(s) = G_\tau(s)\,\tau(s) = \frac{r}{sJ + b_1 + b_2}\,\tau(s)}$$

let $\tau(t) = \tilde{\tau}\cos(\omega t)$, $t \geq 0$, then the steady-state response can be computed using freq. response:

for $\tau(t) = \tilde{\tau}\cos(\omega t) \Rightarrow V_{1,ss}(t) = \tilde{\tau}\,|G_\tau(j\omega)|\cos(\omega t + \measuredangle G_\tau(j\omega))$

where:

$$|G_\tau(j\omega)| = \left|\frac{r}{(j\omega)J + b_1 + b_2}\right| = \frac{r}{\sqrt{(J\omega)^2 + (b_1+b_2)^2}}$$

$$\measuredangle G_\tau(j\omega) = \measuredangle \frac{r}{(j\omega)J + b_1 + b_2} = \measuredangle r - \measuredangle (j\omega)J + b_1 + b_2 = 0 - \tan^{-1}\left(\frac{J\omega}{b_1+b_2}\right)$$

$$\Rightarrow \boxed{V_{1,ss}(t) = \tilde{\tau}\,\frac{r}{\sqrt{(J\omega)^2 + (b_1+b_2)^2}}\cos\left(\omega t - \tan^{-1}\left(\frac{J\omega}{b_1+b_2}\right)\right)} \quad (*)$$

Comparing (*) w/ the answer of P 3.23 which is recalled as follows:

$$v_{1,ss} = \frac{r\tilde{\tau}}{(b_1+b_2)^2 + J^2\omega^2}\left((b_1+b_2)\cos(\omega t) + J\omega\sin(\omega t)\right) \quad (**)$$

we note that (*) is the compact form of (**) and thus is easier & faster to be computed.

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P 3.25:

we recall the dynamics that is provided in the problem statement:

$$J\dot{\omega} + \left(b + \frac{k_t k_e}{R_a}\right)\omega = \frac{k_t}{R_a} v_a$$

we then compute the transfer fcn from v_a to ω as follows:

$$\mathcal{L}\left\{J\dot{\omega} + \left(b + \frac{k_t k_e}{R_a}\right)\omega = \frac{k_t}{R_a} v_a\right\} \Rightarrow \mathcal{L}\left\{J\dot{\omega} + \left(b + \frac{k_t k_e}{R_a}\right)\omega\right\} = \mathcal{L}\left\{\frac{k_t}{R_a} v_a\right\}$$

by linearity property of laplace:

$$\mathcal{L}\{J\dot{\omega}\} + \mathcal{L}\left\{\left(b + \frac{k_t k_e}{R_a}\right)\omega\right\} = \mathcal{L}\left\{\frac{k_t}{R_a} v_a\right\}$$

$$\Rightarrow sJ\,W(s) - J\omega(0) + \left(b + \frac{k_t k_e}{R_a}\right) W(s) = \frac{k_t}{R_a} V_a(s)$$

Assuming zero-initial conditions:

$$\left(sJ + \left(b + \frac{k_t k_e}{R_a}\right)\right) W(s) = \frac{k_t}{R_a} V_a(s) \Rightarrow G(s) = \frac{W(s)}{V_a(s)} = \frac{\frac{k_t}{R_a}}{sJ + \left(b + \frac{k_t k_e}{R_a}\right)}$$

$$\Rightarrow \boxed{G(s) = \frac{W(s)}{V_a(s)} = \frac{k_t}{(R_a J)s + (bR_a + k_t k_e)}} \quad (*)$$

the only pole of $(*)$ is $p = -\frac{bR_a + k_t k_e}{R_a J}$ which assuming all the parameters are positive is negative & thus the transfer fcn $G(s)$ is asymptotically stable.

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P 3.27:

We recall $G(s)$ from the previous problem:

$$G(s) = \frac{\omega(s)}{V_a(s)} = \frac{K_t}{(R_a J)s + (bR_a + K_t + K_e)}$$

then w/ $v_a(t) = \tilde{v}_a \cos(\omega t),\ t \geq 0$, the steady state response using freq response technique can be obtained as follows:

$$\omega_{ss}(t) = \tilde{v}_a |G(j\omega)| \cos(\omega t + \measuredangle G(j\omega)) \quad (*)$$

where

$$|G(j\omega)| = \left| \frac{K_t}{(R_a J)(j\omega) + (bR_a + K_t + K_e)} \right| = \frac{K_t}{\sqrt{(R_a J\omega)^2 + (bR_a + K_t + K_e)^2}} \quad (**)$$

and

$$\measuredangle G(j\omega) = \measuredangle \frac{K_t}{(R_a J)(j\omega) + (bR_a + K_t + K_e)} = \measuredangle K_t - \measuredangle (R_a J) j\omega + (bR_a + K_t + K_e)$$

$$\Rightarrow \measuredangle G(j\omega) = -\tan^{-1}\left(\frac{R_a J\omega}{bR_a + K_t + K_e}\right) \quad (***)$$

(*) by (**) & (***) yields:

$$\boxed{\omega_{ss}(t) = \tilde{v}_a \frac{K_t}{\sqrt{(R_a J\omega)^2 + (bR_a + K_t + K_e)^2}} \cos\left(\omega t - \tan^{-1}\left(\frac{R_a J\omega}{bR_a + K_t + K_e}\right)\right)}$$

END of HW# 4 Sol'n