Problems

P5.1. Write state-space equations for the block diagrams in Fig. 5.17 and in Fig. 5.18 and compute the transfer-function from u to y.

P5.2. You have shown in **P2.1** that the ordinary differential equation

$$m\,\dot{v} + b\,v = m\,g$$

is a simplified description of the motion of an object of mass m dropping vertically under constant gravitational acceleration, g, and linear air resistance, $-b\,v$. Let the gravitational force, mg, be the input and the vertical velocity, v, be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.3. Repeat **P5.2** considering the vertical position, $x = \int_0^t v(\tau)\,d\tau$, as the output.

P5.4. Repeat **P5.2** considering the vertical acceleration, $\dot{v}$, as the output. *Hint: Use the original equation to obtain $\dot{v}$ as a function of v.*

P5.5. The ordinary differential equation

$$m\,\dot{v} + b\,v|v| = m\,g$$

is a simplified description of the motion of an object of mass m dropping vertically under constant gravitational acceleration, g, and quadratic air resistance, $-b\,v|v|$. Let the gravitational force, mg, be the input and the vertical velocity, v, be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.6. Use the block-diagram obtained in **P5.5** to simulate the velocity of an object with $m =$ 70kg, $b = 0.0175$kg/m, and $g = 10$m/s^2 falling with zero initial velocity. Compare your solution with the one given in **P2.5**.

P5.7. Use **P5.5** to redo **P2.4**.

P5.8. Calculate the equilibrium points of the state-space representation obtained in **P5.5**. Linearize the state-space equations about the equilibrium points and compute the corresponding transfer-functions. Are the equilibrium points asymptotically stable?

P5.9. You have shown in **P2.7** that the ordinary differential equation

$$(J_1 r_1^2 + J_2 r_2^2)\,\dot{\omega}_1 = r_2^2\,\tau,$$
$$\omega_2 = (\,r_1/r_2\,)\,\omega_1$$

is a simplified description of the motion of a rotating machine driven by a belt without slip as in Fig. 2.18, where ω_1 is the angular velocity of the driving shaft and ω_2 is the machine's angular velocity. Let the torque, τ, be the input and the machine's angular velocity, ω_2, be the output

Fig. 5.17: Block diagrams for **P5.1**

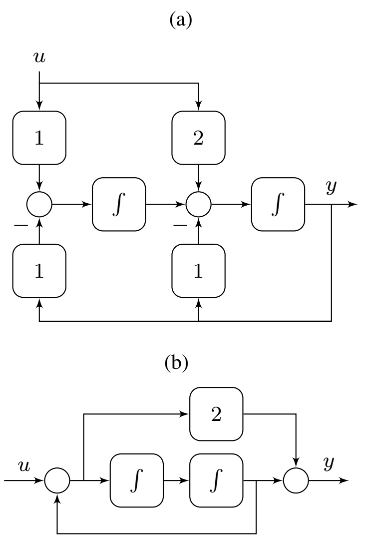
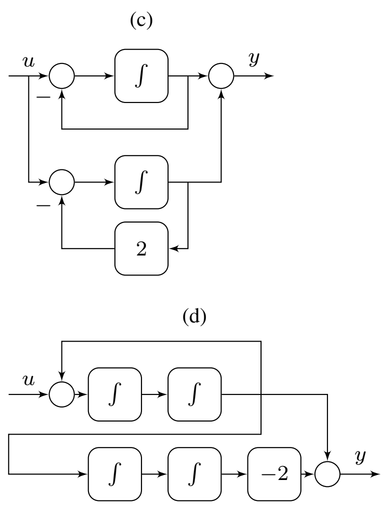


Fig. 5.18: Block diagrams for **P5.1**



and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.10. Repeat **P5.9** considering the machine's angle, $\theta_2 = \int_0^t \omega_2(\tau)\, d\tau$, as the output.

P5.11. Repeat **P5.9** considering the machine's angular acceleration, $\dot{\omega}_2$, as the output. *Hint: Use the original equation to obtain $\dot{\omega}_2$ as a function of ω_1.*

P5.12. You have shown in **P2.10** that the ordinary differential equation

$$\begin{aligned} J\,\dot{\omega} + (b_1 + b_2)\,\omega &= \tau + g\,r(m_1 - m_2), \\ J &= J_1 + J_2 + r^2(m_1 + m_2), \\ v_1 &= r\,\omega, \end{aligned}$$

is a simplified description of the motion of the elevator in Fig. 2.19, where ω is the angular velocity of the driving shaft and v_1 is the elevator's load linear velocity. Let the torque, τ, and the gravitational torque, $g\,r(m_1 - m_2)$, be inputs and the elevator's linear velocity, v_1, be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.13. Compute the transfer-function from the two inputs to the single output in **P5.12**. The transfer-function will be a 1×2 matrix. *Hint: Use the state-space equations and equation* (5.6).

P5.14. You have shown in **P2.16** that the ordinary differential equation

$$J\,\dot{\omega} + \left(b + \frac{K_t K_e}{R_a}\right)\omega = \frac{K_t}{R_a} v_a$$

is a simplified description of the motion of the rotor of the DC motor in Fig. 2.20, where ω is the rotor angular velocity. Let the armature voltage, v_a, be the input and the angular velocity, ω, be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.15. Repeat **P5.14** considering the rotor's angle, $\theta_2 = \int_0^t \omega_2(\tau)\, d\tau$, as the output.

P5.16. Recall from **P2.16** that the rotor torque:

$$\tau = K_t\, i_a.$$

The armature current, i_a, is related to the armature voltage, v_a, and the rotor angular velocity, ω, through

$$v_a = R_a\, i_a + K_e\, \omega.$$

Repeat **P5.14** considering the rotor torque as the output. At what (constant) angular velocity the motor attains its highest torque?

P5.17. You have shown in **P2.31** that the ordinary differential equation

$$R C_2\, \dot{v}_o + R C_1\, \dot{v}_i + v_i = 0.$$

is an approximate model for the electric circuit in Fig. 2.27. Let the input voltage, v_i, be the input and the output voltage, v_o, be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equation in state-space form.

P5.18. You have shown in **P2.28** that the ordinary differential equations:

$$\begin{aligned} &m_1\ddot{x}_1 + (b_1 + b_2)\dot{x}_1 + \\ &\qquad (k_1 + k_2)x_1 - b_2\dot{x}_2 - k_2 x_2 = 0, \\ &m_2\ddot{x}_2 + b_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = f_2 \end{aligned}$$

constitute a simplified description of the motion of the mass-spring-damper system in Fig. 2.25 where x_1 and x_2 are displacements and f_2 is a force applied on the mass m_2. Let the force, f_2, be the input and the displacement, x_2, be the output and represent this equation in a block-diagram using only integrators. Rewrite the differential equations in state-space form.

P5.19. Let $m_1 = m_2 = 1$kg, $b_1 = b_2 = 0.1$kg/s, $k_1 = 1$N/m, $k_2 = 2$N/m. Use MATLAB to compute the transfer-function from the force f_2 to the displacement x_2. Is this system asymptotically stable?

P5.20. The equations of motion of a rigid-body with principal moments of inertia J_1, J_2 and J_3 is given by Euler's equations:

$$\begin{aligned} J_1\dot{\omega}_1 + \omega_2\omega_3(J_3 - J_2) &= 0, \\ J_2\dot{\omega}_2 + \omega_3\omega_1(J_1 - J_3) &= 0, \\ J_3\dot{\omega}_3 + \omega_3\omega_2(J_2 - J_1) &= 0, \end{aligned}$$

where the angular velocity vector

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

is the state vector. Verify that

$$\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3) = (\Omega, 0, 0)$$

is an equilibrium point. Linearize the equations about this equilibrium point. Show that if $J_2 < J_1 < J_3$ or $J_3 < J_1 < J_2$ then this is an unstable equilibrium point. Interpret this result.

P5.21. Compute the linearized equations for the pendulum in a cart model developed in § 5.5. Let $m_p = 2$kg, $m_c = 10$kg, $\ell = 1$m, $b_p = 0.01$kg m^2/s, $b_c = 0.1$km/s, $r = \ell/2$, $J_p = m\ell^2 \big/ 12$, and use MATLAB to compute the transfer-function from u to θ and from u to $\dot{x}_c$ around the equilibrium points calculated with $\bar{\theta} = 0$ and $\bar{\theta} = \pi$ and $\bar{u} = 0$. Are the equilibria asymptotically stable?

P5.22. We have shown in § 2.8 that the water level, h, in a rectangular water tank of cross-section area A can be modeled as the integrator:

$$\dot{h} = \frac{1}{A} w_{\text{in}}$$

where w_{in} is the in-flow rate. If water is allowed to flow out from the bottom of the tank through an orifice then

$$\dot{h} = \frac{1}{A} \left(w_{\text{in}} - w_{\text{out}} \right).$$

The out-flow rate can be approximated by

$$w_{\text{out}} = \frac{1}{R} \left(p_{\text{t}} - p_{\text{a}} \right)^{1/\alpha}$$

where the *resistance* $R > 0$ and the exponent α depend on the shape of the out-flow orifice, p_{a} is the ambient pressure outside the tank, and

$$p_{\text{t}} = p_{\text{a}} + \rho\, g\, h,$$

is the pressure at the water level, ρ is the water density and g is the gravitational acceleration. Combine these equations to write a nonlinear differential equation in state-space relating the water in-flow rate w_{in} with the water tank level, h. Determine a water in-flow rate w_{in} such that the tank system is in equilibrium with a water level $h = \bar{h} > 0$. Linearize the state-space equations about this equilibrium point for $\alpha = 2$ and compute the corresponding transfer-function. Is this equilibrium point asymptotically stable?