# Problems

**P6.1**. Show that the roots of the second-order polynomial

$$s^2 + 2\zeta\omega_n s - \omega_n^2 = 0, \qquad \omega_n > 0.$$

are always real and at least one of the roots is positive.

**P6.2**. Show that the step response of the second-order system with transfer-function

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \omega_n > 0$$

is

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \pi/2 - \phi_d),$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2},$$

$$\phi_d = \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}},$$

for $t \geq 0$.

**P6.3**. The mass-spring-damper diagram in Figure 6.26 is used to model the suspension dynamics of a car [Jaz08, Chapter 14]. The mass $m$ represents $1/4$ of the mass of the car and $m_u$ represents the mass of the wheel. The constants $k$ and $b$ are the stiffness and damping coefficient of the spring and shock absorber. The ordinary differential equation:

$$m\ddot{x} + b\dot{x} + kx = ky + b\dot{y}$$

constitute a simplified description of the motion of this model where $x$ is a displacement measured from equilibrium. If

$$z = x - y,$$

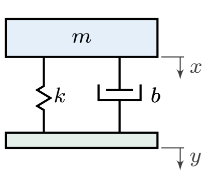
show that

$$m\ddot{z} + b\dot{z} + kz = -m\ddot{y}$$

where $y$ is the road profile.

**P6.4**. Calculate the transfer-function from the road profile $y$ to the displacement $z$. Calculate

Fig. 6.24: One-eight-car model, **P6.3**



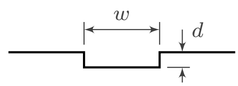




Fig. 6.25: Diagram for **P6.5**

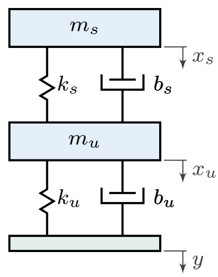



Fig. 6.26: One-quarter car model, **P6.7**

the value of the spring stiffness and shock absorber damping coefficient for a car with $1/4$ mass equal to 640kg to have a natural frequency $f_n = 10$Hz and damping ratio $\zeta = 0.08$. Locate the roots in the complex plane.

**P6.5**. Use MATLAB to plot the response of the car suspension with parameters as in **P6.4** to a pothole with a profile as shown in Fig. 6.25 where $w$ =1m and $d$ = 5cm for a car traveling at 10km/h. Repeat for a car traveling at 100km/h. Comment on your findings.

**P6.6**. If the road profile is $y(t) = \cos(\lambda v t)$ where $v$ is the car's velocity, what is the worst possible velocity a car with suspension parameters as in **P6.4** can be traveling as a function of the road wavelength $\lambda$?

**P6.7**. The spring-mass-damper diagram in Figure 6.26 is used to model the wheel-body dynamics of a car [Jaz08, Chapter 15] and is an improved version of the model considered in **P6.3**. The mass $m_s$ represents $1/4$ of the mass of the car without the wheels and $m_u$ represents the mass of a single wheel. The constants $k_s$ and $b_s$ are the stiffness and damping coefficient of the spring and shock absorber. The constants $k_u$ and $b_u$ are the stiffness and damping coefficient of the tire. The ordinary differential equations:

$$m_s\ddot{x}_s + b_s(\dot{x}_s - \dot{x}_u) + k_s(x_s - x_u) = 0$$

$$m_u\ddot{x}_u + (b_u + b_s)\dot{x}_u + (k_u + k_s)x_u - b_s\dot{x}_s - k_s x_s = k_u y + b_u\dot{y}$$

constitute a simplified description of the motion of the quarter-car model where $x_s$ and $x_u$ are displacements measured from equilibrium. If

$$x = x_s - x_u, \qquad z = x_u - y,$$

show that

$$\begin{aligned} &m_s m_u \ddot{x} + b_s(m_s + m_u)\dot{x} + \\ &\qquad k_s(m_s + m_u)x - \\ &\qquad\qquad k_u m_s z - b_u m_s \dot{z} = 0 \\ &m_u \ddot{z} + k_u z + b_u \dot{z} - k_s x - b_s \dot{x} = -m_u \ddot{y} \end{aligned}$$

where $y$ is the road profile.

**P6.8**. Calculate the transfer-function from the road profile $y$ to the displacement $x + z$. If the tire stiffness is $k_u = 200000$N/m and the tire damping coefficient is negligible, i.e. $b_u = 0$, use MATLAB to calculate the value of the spring stiffness and shock absorber damping coefficient for a car with $1/4$ mass $m_s$ equal to 600kg and wheel mass equal to $m_u = 40$kg to have its dominant poles have a natural frequency $f_n = 10$Hz and damping ratio $\zeta = 0.08$. Locate all the roots in the complex plane.

**P6.9**. Use MATLAB to plot the response of the car suspension with parameters as in **P6.8** to a pothole with a profile as shown in Fig. 6.25 where $w$ =1m and $d$ = 5cm for a car traveling at 10km/h. Repeat for a car traveling at 100km/h. Comment on your findings.

**P6.10**. If the road profile is $y(t) = \cos(vt/\lambda)$ where $v$ is the car's velocity, what is the worst possible velocity a car with suspension parameters as in **P6.8** can be traveling as a function of the road wavelength $\lambda$?

**P6.11**. Compare the answers from **P6.8–P6.10** with the answers from **P6.4–P6.4**.

**P6.12**. We have shown in § 5.4 that the nonlinear differential equation:

$$J_r\, \ddot{y} + b\, \dot{y} + m\, g\, r \sin\theta = u$$

is an approximate model for the motion of the simple pendulum in Fig. 5.11 and that $(\bar{\theta}, \bar{u}) = (0, 0)$ and $(\bar{\theta}, \bar{u}) = (\pi, 0)$ are the pendulum equilibrium points. Calculate the nonlinear differential equation obtained in closed-loop with the linear proportional controller:

$$u = K\, e, \qquad e = \bar{\theta} - \theta.$$

Show that $(\bar{\theta}, \bar{u}) = (0, 0)$ and $(\bar{\theta}, \bar{u}) = (\pi, 0)$ are still equilibrium points. Linearize the closed-loop system linearized about $(\bar{\theta}, \bar{u}) = (0, 0)$

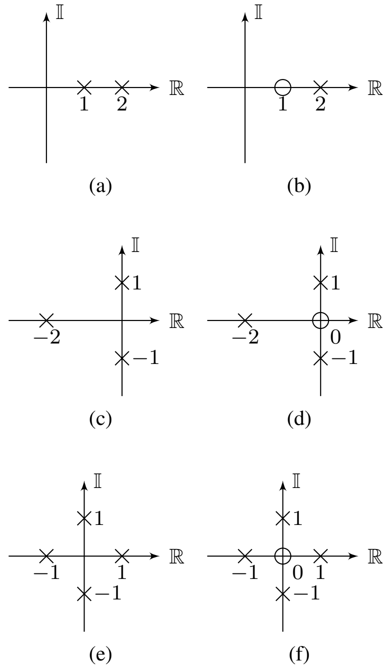



Fig. 6.27: Block diagrams for **P6.14**

and $(\bar{\theta}, \bar{u}) = (\pi, 0)$ and calculate the associated transfer-function. Assuming all constants are positive, find the range of values of $K$ that stabilize both equilibrium points.

**P6.13**. Repeat **P6.12** with the linear proportional-derivative controller:

$$u = K_p\, e - K_d\, \dot{e}, \qquad e = \bar{\theta} - \theta.$$

**P6.14**. First sketch the root-locus for the SISO systems with poles and zeros shown in Fig. 6.27 then use MATLAB to verify your answer.

**P6.15**. Find a proper feedback controller that can stabilize the SISO systems with poles and zeros shown in Fig. 6.27. Is the transfer-function of the controller asymptotically stable?

**P6.16**. You have shown in **P2.7** and **P2.9** that the ordinary differential equation

$$J_r\, \dot{\omega}_1 + b_r\, \omega_1 = r_2^2\, \tau, \quad \omega_2 = (\, r_1/r_2\,)\, \omega_1,$$

is a simplified description of the motion of a rotating machine driven by a belt without slip as in Fig. 2.18, where

$$J_r = J_1 r_2^2 + J_2 r_1^2, \quad b_r = b_1 r_2^2 + b_2 r_1^2,$$

$\omega_1$ is the angular velocity of the driving shaft and $\omega_2$ is the machine's angular velocity. Let $r_1 = 0.05$m, $r_2 = 0.25$m, $m_1 = 1$kg, $m_2 =$ 10kg, $b_1 = 0.125$kg m/s, $b_2 = 6.25$kg m/s, $J_i = m_i r_i^2/2$, $i = 1, 2$. Use the root-locus method to design an I-controller:

$$\tau = K \int_0^t e(\sigma)\, d\sigma, \qquad e = \bar{\omega}_2 - \omega_2$$

and select $K$ so that both closed-loop poles are real and as negative as possible. Is the closed-loop capable of asymptotically tracking a constant reference input $\bar{\omega}_2$?

**P6.17**. Redo **P6.16** using a PI-controller and select the gains such that both closed-loop poles have negative real part more negative than the pole of the machine without performing a pole-zero cancellation. Is the closed-loop capable of asymptotically tracking a constant reference input $\bar{\omega}_2$?

**P6.18**. The belt system in **P6.16** is connected to a piston that applies a periodic torque that can be approximated by $\tau_2(t) = h\cos(\sigma t)$, where the angular frequency $\sigma$ is equal to the angular velocity $\omega_2$. The modified equations including this additional torque is given by:

$$J_r\, \dot{\omega}_1 + b_r\, \omega_1 = r_2(r_2\, \tau_1 + r_1 \tau_2).$$

Use the root-locus method to design a controller so that the closed-loop system is capable of asymptotically tracking a constant reference input $\bar{\omega}_2(t) = \bar{\omega}_2 = 4\pi$, $t \geq 0$, and asymptotically rejecting the torque perturbation $\tau_2(t) = h\cos(\sigma t)$ when $\sigma = \bar{\omega}_2$.

**P6.19**. You have shown in **P2.10** that the ordinary differential equation

$$J\, \dot{\omega} + (b_1 + b_2)\, \omega = \tau + g\, r(m_1 - m_2),$$
$$J = J_1 + J_2 + r^2(m_1 + m_2),$$
$$v_1 = r\, \omega,$$

is a simplified description of the motion of the elevator in Fig. 2.19, where $\omega$ is the angular velocity of the driving shaft and $v_1$ is the elevator's load linear velocity. Let $r = 1$m, $m_1 =$ 1000kg, $m_2 = 800$kg, $b_1 = b_2 = 120$kg m/s, $J_1 = J_2 = 200$kg m$^2$, and $g = 10$m/s$^2$. Use the root-locus method to design a dynamic controller $\tau = K(\bar{x}_1 - x_1)$ that can asymptotically track a constant position reference $\bar{x}_1(t) = \bar{x}_1$, $t \geq 0$, where $x_1 = \int_0^t v_1(\tau)\, d\tau$ is the elevator's load linear position.

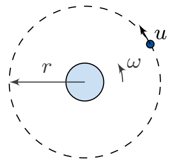



Fig. 6.28: Satellite in orbit, **P6.20**

**P6.20**. The differential equations:

$$m(\ddot{r} - r\,\omega^2) = -\frac{GMm}{r^2}$$
$$m(2\,\dot{r}\,\omega + r\,\dot{\omega}) = u$$

are a simplified description of the motion of a satellite orbiting earth as in Fig. 6.28, where $r$ is the satellite's radial distance from the center of the earth and $\omega$ is the satellite's angular velocity, $u$ is the force applied by a thruster in the tangential direction, $m$ is the mass of the satellite, $M \approx 6 \times 10^{24}$kg is the mass of the earth, and $G \approx 6.7 \times 10^{-11}$Nm$^2$/kg$^2$ is the universal gravitational constant. Represent this equation in a block-diagram using only integrators and rewrite the differential equations in state-space.

**P6.21**. Show that if

$$\Omega^2 R^3 = GM$$

then $u(t) = \dot{r}(t) = 0$, $r(t) = R$ and $\omega(t) = \Omega$ is an equilibrium point of the equations in **P6.20**. Linearize the equations about the equilibrium point to obtain the linearized system in state space:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 3\Omega^2 & 0 & 2\Omega \\ 0 & -2\Omega & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m} \end{bmatrix} u,$$
$$\tilde{y} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x,$$

where

$$x = \begin{pmatrix} r - R \\ \dot{r} \\ R\omega - R\Omega \end{pmatrix}, \qquad \tilde{y} = y - R.$$

Use MATLAB to calculate the transfer-function from the tangential thrust $u$ to the output $\tilde{y}$ and design a controller using the root-locus method that can stabilize the radial distance of the satellite in closed-loop.