MAE 143B - Homework 7

5.1

a)

Let x_1 and x_2 be the outputs of the first and second integrator, respectively. Omitting time-dependencies, we immediately get the equations

$$\begin{aligned} y &= x_2, \\ \dot{x}_1 &= u - y = u - x_2, \\ \dot{x}_2 &= 2u - y + x_1 = 2u + x_1 - x_2. \end{aligned}$$

Writing these equations in matrix form results in the state-space representation

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u, \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u. \end{aligned}$$

Using the formula derived in class, the corresponding transfer function is

$$G(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & 1 \\ -1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{s(s+1)+1} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & -1 \\ 1 & s \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{2s+1}{s^2+s+1}.$$

b)

By the same approach followed in the previous case, we get

$$\begin{aligned} y &= x_2 + 2(u + x_2) = 3x_2 + 2u, \\ \dot{x}_1 &= u + x_2, \\ \dot{x}_2 &= x_1, \end{aligned}$$

which in state-space form reads

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 2 \cdot u. \end{aligned}$$

The transfer function is

$$G(s) = \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 = \frac{1}{s^2-1} \begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 = \frac{2s^2+1}{s^2-1}.$$

c)

As before, we get

$$\begin{aligned} y &= x_1 + x_2. \\ \dot{x}_1 &= u - x_1, \\ \dot{x}_2 &= u - 2x_2, \end{aligned}$$

such that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u$$

and

$$G(s) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2s+3}{(s+1)(s+2)}.$$

5.18

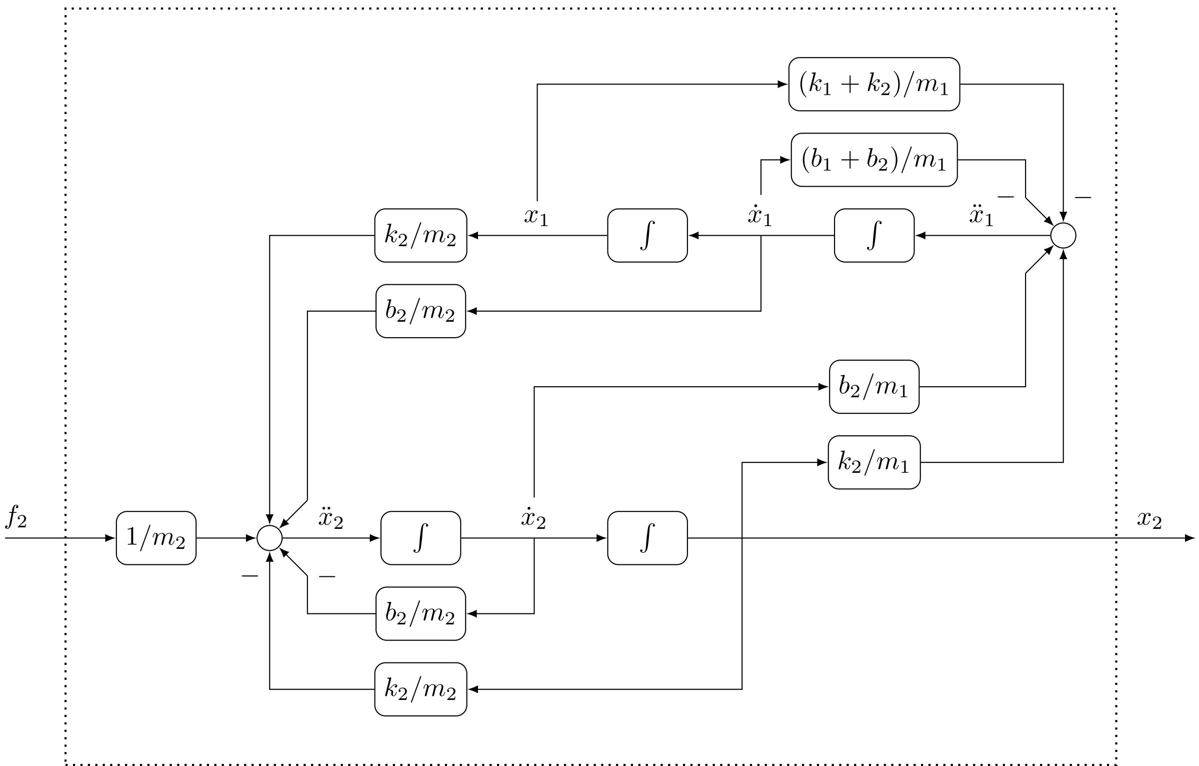

Figure 1: Block Diagram for 5.18

Have a look at Figure 1 for a block diagram representing the given equations. Denoting $z_1 = x_1$, $z_2 = \dot{x}_1$, $z_3 = x_2$, $z_4 = \dot{x}_2$, $y = x_2$ and $u = f_2$, we get the equations

$$\begin{aligned}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= -\frac{k_1+k_2}{m_1} z_1 - \frac{b_1+b_2}{m_1} z_2 + \frac{k_2}{m_1} z_3 + \frac{b_2}{m_1} z_4, \\
\dot{z}_3 &= z_4, \\
\dot{z}_4 &= \frac{k_2}{m_2} z_1 + \frac{b_2}{m_2} z_2 - \frac{k_2}{m_2} z_3 - \frac{b_2}{m_2} z_4 + \frac{1}{m_2} u.
\end{aligned}$$

In matrix form, this results in the state-space representation

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(k_1+k_2)/m_1 & -(b_1+b_2)/m_1 & k_2/m_1 & b_2/m_1 \\ 0 & 0 & 0 & 1 \\ k_2/m_2 & b_2/m_2 & -k_2/m_2 & -b_2/m_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + 0 \cdot u.$$

5.19

With the given parameters, we can refine our state-space model to obtain (omitting physical units here)

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & -0.2 & 2 & 0.1 \\ 0 & 0 & 0 & 1 \\ 2 & 0.1 & -2 & -0.1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + 0 \cdot u.$$

We can now evaluate our formula in MATLAB:

```
A = [0 1 0 0; -3 -0.2 2 0.1;0 0 0 1;2 0.1 -2 -0.1];
B = [0;0;0;1];
C = [0 0 1 0];
D = 0;

[num,den] = ss2tf(A,B,C,D);
G = tf(num,den);
```

The last line should give you the result

$$G(s) = \frac{s^2 + 0.2s + 3}{s^4 + 0.3s^3 + 5.01s^2 + 0.3s + 2},$$

no matter how you constructed your state-space representation. The reason for using the MATLAB commands `tf` and `ss2tf` instead of direct application of our formula are potential pole-zero cancellations. That is, whenever we have coinciding poles and zeros, these would be cancelled by MATLAB without our knowledge. However, as you have seen in class, this may mask important properties of our system. The zeros and poles of the above transfer function can be computed as following:

```
>> p = roots(den)

p =

  -0.1296 + 2.1317i
  -0.1296 - 2.1317i
  -0.0204 + 0.6619i
  -0.0204 - 0.6619i

>> z = roots(num)

z =
```

```
  -0.1000 + 1.7292i
  -0.1000 - 1.7292i
```

All poles having strictly negative real parts, the system is asymptotically stable. Moreover, no pole-zero cancellations did occur.

5.22

Combing our equations, we get

$$w_{out} = \frac{1}{R}\left(p_t - p_a\right)^{1/\alpha} = \frac{1}{R}\left(\rho g h\right)^{1/\alpha},$$

yielding the first order nonlinear differential equation

$$\dot{h} = \frac{1}{A}(w_{in} - w_{out}) = \frac{1}{A}\left(w_{in} - \frac{1}{R}\left(\rho g h\right)^{1/\alpha}\right).$$

The system is in equilibrium whenever $\dot{h} = 0$, which at $\bar{h} > 0$ holds for

$$w_{in} = \frac{1}{R}\left(\rho g \bar{h}\right)^{1/\alpha} =: \bar{w}_{in}.$$

To linearize the system for $\alpha = 2$ at $\bar{h}$, we evaluate the partial derivatives

$$\mathbf{A} = \frac{\partial}{\partial h}\dot{h}\bigg|_{\bar{w}_{in},\bar{h}} = -\frac{\rho g}{2AR}\cdot\frac{1}{(\rho g\bar{h})^{1/2}} = -\frac{\sqrt{\rho g}}{2AR\sqrt{\bar{h}}} \qquad \text{and} \qquad \mathbf{B} = \frac{\partial}{\partial w_{in}}\dot{h}\bigg|_{\bar{w}_{in},\bar{h}} = \frac{1}{A}.$$

Moreover, we define state $x := h - \bar{h}$, output $y := h - \bar{h} = x$ and control input $u := w_{in} - \bar{w}_{in}$, such that the linearized system reads

$$\begin{aligned}\dot{x} &= -\frac{\sqrt{\rho g}}{2AR\sqrt{\bar{h}}}x + \frac{1}{A}u,\\ y &= x.\end{aligned}$$

Using our formula, the transfer function is

$$G(s) = \frac{1}{s + \frac{\sqrt{\rho g}}{2AR\sqrt{\bar{h}}}}\cdot\frac{1}{A} = \frac{A^{-1}}{s + \frac{\sqrt{\rho g}}{2AR\sqrt{\bar{h}}}},$$

which has a single pole at $s = \mathbf{A} = -\frac{\sqrt{\rho g}}{2AR\sqrt{\bar{h}}}$. By $\bar{h} > 0$, $R > 0$ and physical interpretation of the remaining constants, this pole has strictly negative real part, implying that our transfer function and thus the equilibrium point are asymptotically stable.