P6.3:

Given the following ODE:

$$m\ddot{x} + b\dot{x} + kx = ky + b\dot{y} \quad (*)$$

and that $z = x - y$, we shall substitute for x in $(*)$ w/ $x = z + y$. Therefore, we obtain:

$$m(\ddot{z} + \ddot{y}) + b(\dot{z} + \dot{y}) + k(z + y) = ky + b\dot{y}$$

which then yields:

$$m\ddot{z} + m\ddot{y} + b\dot{z} + \cancel{b\dot{y}} + kz + \cancel{ky} = \cancel{ky} + \cancel{b\dot{y}}$$

where then by some simplifications results:

$$m\ddot{z} + m\ddot{y} + b\dot{z} + kz = 0 \rightarrow \boxed{m\ddot{z} + b\dot{z} + kz = -m\ddot{y}}$$

P6.4:

Based on the final result derived in P6.3, and w/ zero-initial conditions assumptions, we can obtain the following:

$$m\ddot{z} + b\dot{z} + kz = -m\ddot{y} \xRightarrow{\mathcal{L}} \mathcal{L}\{m\ddot{z} + b\dot{z} + kz\} = \mathcal{L}\{-m\ddot{y}\}$$

Provided m, b and k are constant parameters and by linearity property of Laplace transform, we get:

$$m\mathcal{L}\{\ddot{z}\} + b\mathcal{L}\{\dot{z}\} + k\mathcal{L}\{z\} = -m\mathcal{L}\{\ddot{y}\} \Rightarrow ms^2 Z(s) + bsZ(s) + kZ(s) = -ms^2 Y(s)$$

where then, recalling we are seeking a transfer fcn from y to z, we get:

$$\boxed{\frac{Z(s)}{Y(s)} = \frac{-ms^2}{ms^2 + bs + k}} \quad (*)$$

Then, for a given $m = 640\ kg$, we would like to find proper 'b' and 'k' in order to ensure $f_n = 10\ Hz$ and $\zeta = 0.08$. In order to do so, we first rearrange $(*)$

where we get:

$$\frac{Z(s)}{Y(s)} = \frac{-s^2}{s^2 + {}^{b}/_{m}\, s + {}^{k}/_{m}} \qquad (**)$$

Comparing then the denominator of (**) w/ characteristic polynomial of a second-order system in its canonical form, we get:

$$\omega_n^2 = k/m \quad \text{and} \quad 2\zeta\omega_n = b/m$$

Therefore, to ensure $\omega_n = 2\pi f_n = 20\pi$ and $\zeta = 0.08$ w/ $m = 640$ kg, we need to have:

$$\omega_n^2 = k/m \Rightarrow 400\pi^2 = \frac{k}{640} \Rightarrow k = 640 \times 400\pi^2 \Rightarrow \boxed{k = 2.53 \times 10^6 \ \tfrac{N}{m}} \quad (*)$$

$$2\zeta\omega_n = b/m \Rightarrow 2 \times 0.08 \times 20\pi = b/640 \Rightarrow \boxed{b = 6.43 \times 10^3 \ \tfrac{Ns}{m}} \quad (**)$$

with (*) and (**), the transfer f'n, (**) can be re-written as follows:

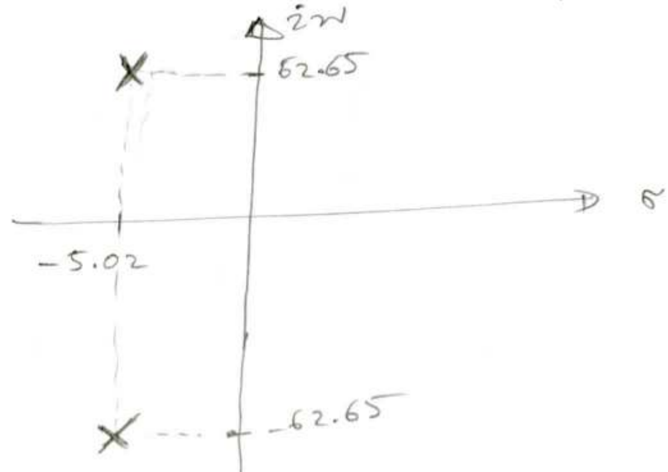
$$\frac{Z(s)}{Y(s)} = \frac{-s^2}{s^2 + \frac{6.43 \times 10^3}{640} s + \frac{2.53 \times 10^6}{640}} \Rightarrow \boxed{\frac{Z(s)}{Y(s)} = \frac{-s^2}{s^2 + 10.05\, s + 3.95 \times 10^3}}$$

where the poles would be:

$$s^2 + 10.05\, s + 3.95 \times 10^3 = 0 \Rightarrow s_{1,2} = \frac{-10.05 \pm \sqrt{(10.05)^2 - 4(3.95 \times 10^3)}}{2}$$

$$\Rightarrow \boxed{s_{1,2} = -5.02 \pm i\, 62.65}$$

The representation of these poles on the complex plane is then:



P6.5:

First we recall the transfer f'n obtained and derived in P6.4 which is as follows:

$$\frac{Z(s)}{Y(s)} = \frac{-s^2}{s^2+2\zeta\omega_n s+\omega_n^2} \ \frac{\zeta=0.08}{\omega_n=20\pi} \ \frac{-s^2}{s^2+2\times 0.08\times 20\pi s+400\pi^2}$$

$$\Rightarrow \frac{Z(s)}{Y(s)} = \frac{-s^2}{s^2+2\times1.6\pi s+2.56\pi^2+397.44\pi^2} \Rightarrow \boxed{\frac{Z(s)}{Y(s)} = \frac{-s^2}{(s+1.6\pi)^2+397.44\pi^2}}$$

(*)

We would like to use (*) to derive Z(s) and z(t) using MATLAB. In order to do so, we need to express Y(s) for which we need y(t), i.e., road profile. This latter can be obtained using Fig. 6.25 and given the car velocity to be $v_1 = 10\,\frac{km}{h}$ and $v_2 = 100\,\frac{km}{h}$, as follows:

- given $w = 1\,m$ and $v_1 = 10\,\frac{km}{h}$, the time that it takes for the car to pass w would be:

$$\left.\begin{array}{l} v_1 = 10\,\frac{km}{h} = 10^4\,\frac{m}{3600 s} \Rightarrow v_1 = \frac{10}{3.6}\,\frac{m}{s} \\ w = 1\,m \end{array}\right\} \Rightarrow T_1 = \frac{w}{v_1} = \frac{1}{\frac{10}{3.6}} \Rightarrow \boxed{T_1 = 0.36\,sec}$$

Having found $T_1 = 0.36 sec$, the road profile can be written as summation of 2 step f'ns w/ $d = 5cm = 0.05m$:

$$\boxed{y_1(t) = -0.05\,u(t) + 0.05\,u(t-0.36)} \quad (**)$$

- given $w = 1m$ and $v_2 = 100\,\frac{km}{h}$, the time that it takes for the car to pass w would be:

$$\left.\begin{array}{l} v_2 = 100\,\frac{km}{h} = 10^5\,\frac{m}{3600 s} \Rightarrow v_2 = \frac{100}{3.6}\,\frac{m}{s} \\ w = 1\,m \end{array}\right\} \Rightarrow T_2 = \frac{w}{v_2} = \frac{1}{\frac{100}{3.6}}$$

$$\Downarrow$$

$$\boxed{T_2 = 0.036 sec}$$

Having found $T_2 = 0.036 sec$, the road profile can be written as summation of 2 step f'ns w/ $d = 5cm = 0.05m$:

$$\boxed{y_2(t) = -0.05\,u(t) + 0.05\,u(t-0.036)} \quad (***)$$

with transfer fcn H(s) and step fcns described in (**) and (***), respectively for $v_1 = 10 \frac{km}{h}$ and $v_2 = 100 \frac{km}{h}$, we can plot the associated responses as shown below.

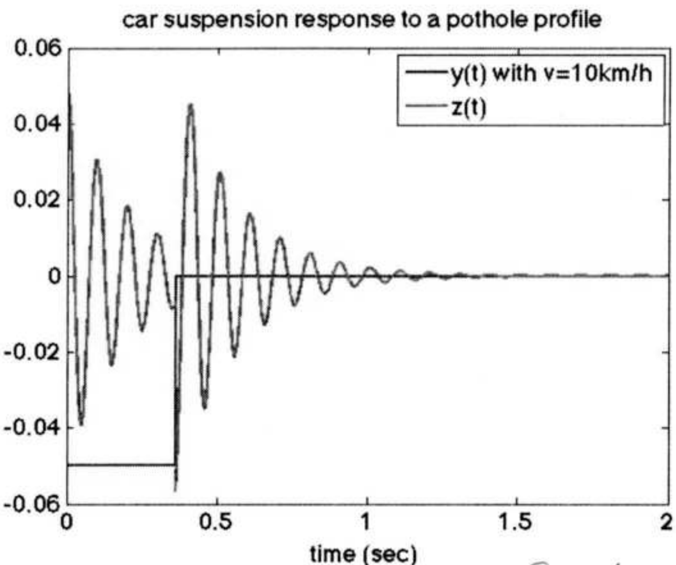


Fig. 1

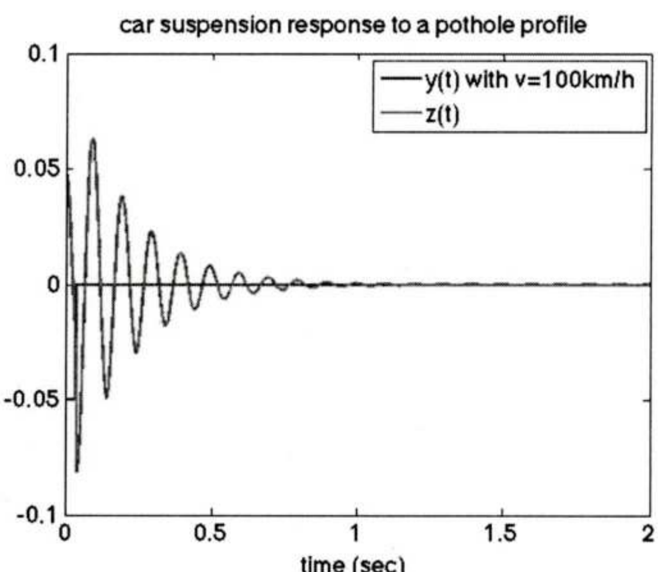


Fig. 2

Comparing fig. 1 and fig. 2, we note that for $v_1 = 10 \frac{km}{h}$ there are 2 distinct excitations whereas for $v_2 = 100 \frac{km}{h}$ there is only one representative excitation. This latter is because of the fact that when the traveling speed is higher the time that it takes to pass the pothole is smaller and thus its excitation is narrower.

P6.14

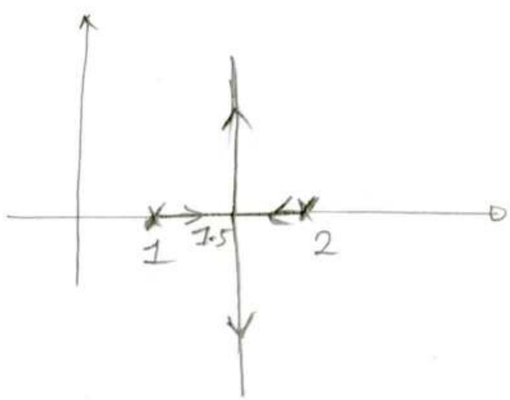
we first draw the root locus using the techniques discussed in the course and then use MATLAB to verify our results.

(a)

Referring to Fig. 6.27 (a), we note that there are two poles and no zeros in the given system, i.e., $m=0$, $n=2$, w/ $P_1 = 1$, $P_2 = 2$. Therefore, according to the root locus rules, we can say the following:

- we start from those 2 poles and we approach the two zeros which are at infinity
- only the segment connecting the two poles on the real axis belong to the root locus and thus no other portion of the real axis
- there would be 2 asymptotes w/ intersecting point of: $C = \frac{1+2-0}{2-0} = 1.5$ and asymptote angles of $\phi_1 = \frac{\pi+0}{2-0} = \pi/2$ and $\phi_2 = \frac{\pi+2\pi}{2-0} = \frac{3\pi}{2}$,

and thus the root locus is sketched as follows:

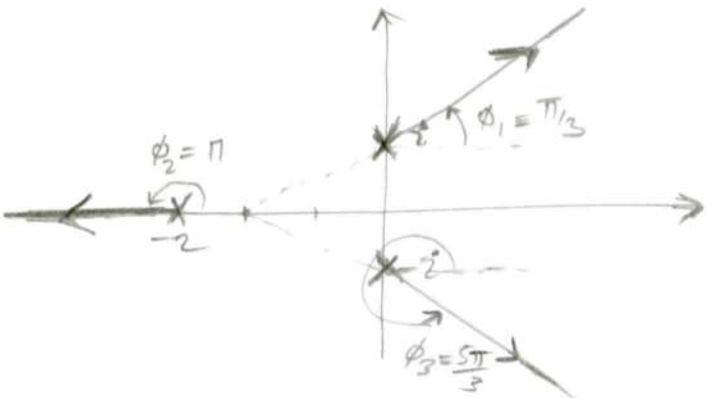


(C)

Referring to Fig. 6.27(C), we note that there are three poles and no zeros in the given system, i.e., $m=0$, $n=3$, w/ $P_1 = i$, $P_2 = -i$, and $P_3 = -2$. Therefore, according to the root locus rules, we can say the following:

- we start from these 3 poles and we approach the 3 zeros which are at infinity
- only the segment connecting $P_3 = -2$ to $-\infty$ on the real axis belong to the root locus and thus no other portion of the real axis.
- there would be 3 asymptotes w/ intersecting point of: $C = \frac{i-i-2-0}{3-0} = -\frac{2}{3}$ and asymptote angles of $\phi_1 = \frac{\pi+0}{3-0} = \frac{\pi}{3}$; $\phi_2 = \frac{\pi+2\pi}{3-0} = \pi$, and $\phi_3 = \frac{\pi+4\pi}{3} = \frac{5\pi}{3}$,

and thus the root locus is obtained as follows:

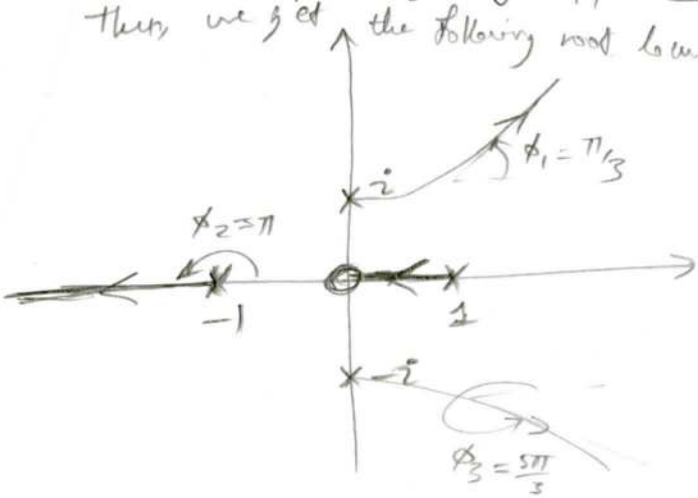


(f)

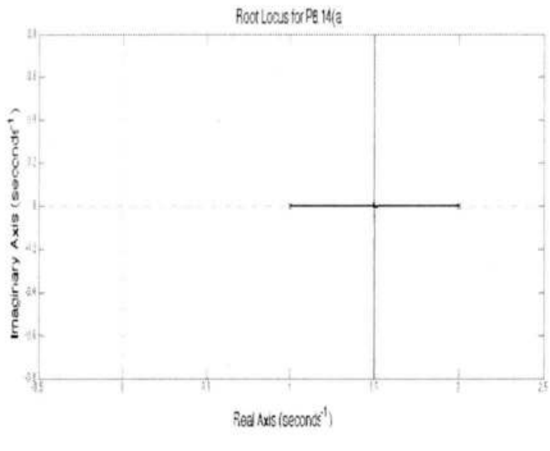
Referring to Fig. 6.27(f), we note that there are four poles and one zero in the given system, i.e., $m = 1$, $n = 4$, w/ $p_1 = i$; $p_2 = -i$, $p_3 = 1$, $p_4 = -1$ and $z_1 = 0$. Therefore, according to the root locus rules, we can say the following:

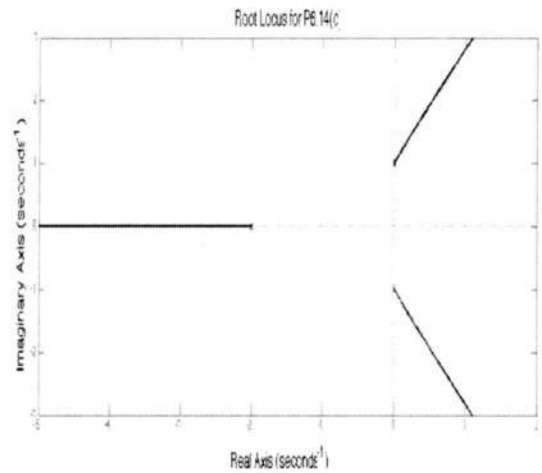
- we start from 4 poles and we approach one finite zero and the other 3 infinite zeros
- only the two segments connecting $p_3 = 1$ to $z_1 = 0$ and $p_4 = -1$ to $-\infty$ belong to the root locus and thus no other portion of the root locus,
- there would be 3 asymptotes w/ intersecting point of: $c = \frac{i - i + 1 - 1 - 0}{3} = 0$ w/ asymptote angles of $\phi_1 = \frac{\pi + 0}{3} = \pi/3$; $\phi_2 = \frac{\pi + 2\pi}{3} = \pi$; $\phi_3 = \frac{\pi + 4\pi}{3} = \frac{5\pi}{3}$.

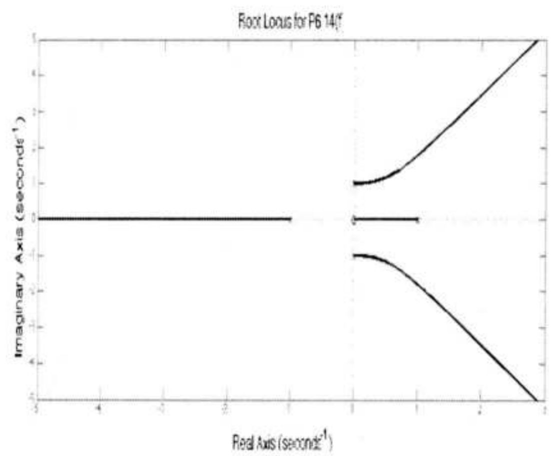
Thus, we get the following root locus for this system:



Then, using MATLAB, the root locus for each part of this problem has been drawn below. page 7







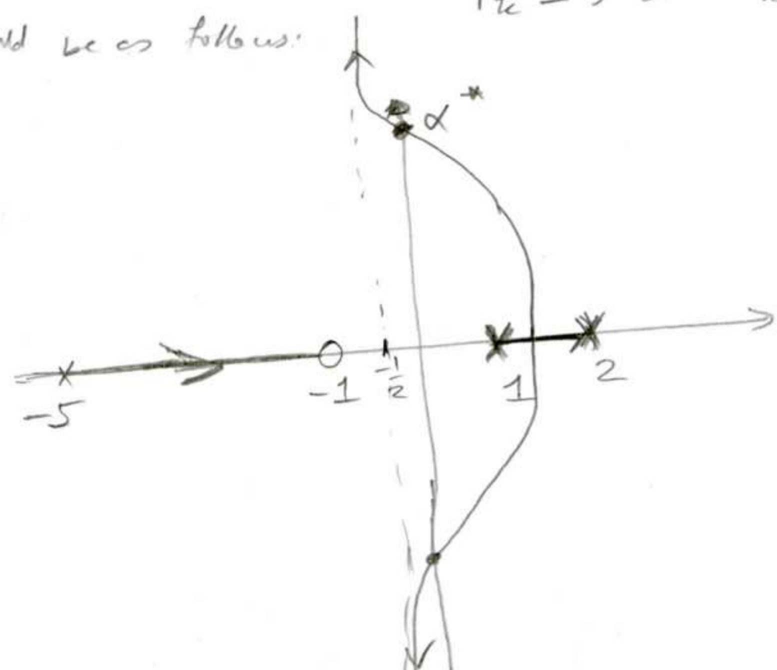
We note that what we had found, based on root locus rules, match very well w/ the above MATLAB-obtained plots.

P6.15:

For each part of this problem, we shall consider controller of the type, $K = \alpha_K \frac{s+z_K}{s+p_K}$, where z_K and p_K are to be located such that at least there would be a value of α_K for which the poles of closed-loop system, $1 + KG = 0$ be located on LHP to ensure the stability.

(a)

For the given set of poles, we shall consider $K = \alpha_K \frac{s+z_K}{s+p_K}$ w/ $z_K > 0$, $p_K > 0$ and $p_K > z_K$, therefore, overall we get 3 poles and one zero, i.e., $m=1$, $n=3$ which yields 2 asymptotes w/ intersecting point, $C = \frac{2+1+(-p_K)-(-z_K)}{3-1} = \frac{3+z_K-p_K}{2}$ and asymptotic angles of $\phi_1 = \frac{\pi}{2}$ and $\phi_2 = \frac{3\pi}{2}$. In fact, to ensure the stability, it is sufficient to ensure that $C < 0$, i.e., $C = \frac{3+z_K-p_K}{2} < 0 \Rightarrow \underline{p_K - z_K > 3}$ this latter is provided $\phi_1 = \pi/2$ and $\phi_2 = 3\pi/2$ that are 2 lines perpendicular to real axis. We have chosen $p_K = 5$ and $z_K = 1$, w/ which the root locus would be as follows:

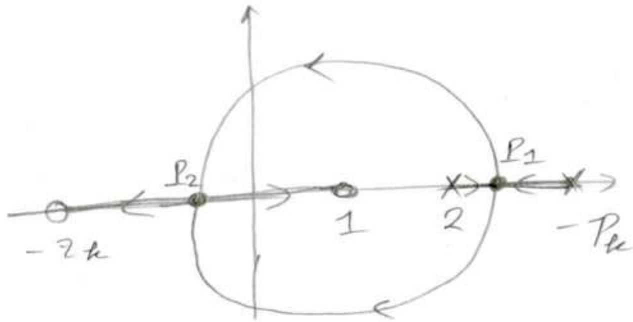


Referring to this fig. we need to find α^* where root locus intersects the imaginary axis; then for $\alpha > \alpha^*$, stability is ensured. Finding α^* is beyond scope of this problem.

Hence, the controller is $K = \alpha_K \frac{s+1}{s+5}$ which is asymptotically stable by its own.

(b)

For the given set of pole and zero, we shall consider $K = \alpha_k \frac{s+z_k}{s+p_k}$ w/ $z_k > 0$, $p_k < 0$ and $|z_k| > |p_k|$, therefore overall we get 2 poles and 2 zeros, $r:0$, $m=n=2$.
In addition, we would like to place z_k and p_k s.t. we get schematically the following root locus:



Therefore, we need to ensure that P_2 is negative. In order to do so, we compute the break away/in points of $KG = \alpha_k \frac{(s+z_k)(s-1)}{(s+p_k)(s-2)}$ as follows:

$$\begin{cases} N(s) = (s+z_k)(s-1) = s^2 - s + z_k s - z_k = s^2 + (z_k - 1)s - z_k \\ D(s) = (s+p_k)(s-2) = s^2 - 2s + p_k s - 2p_k = s^2 + (p_k - 2)s - 2p_k \end{cases}$$

$$\Rightarrow \begin{cases} N'(s) = 2s + (z_k - 1) \\ D'(s) = 2s + (p_k - 2) \end{cases}$$

$$\Rightarrow N(s)D'(s) - N'(s)D(s) = 0 \Rightarrow (s^2 + (z_k-1)s - z_k)(2s + (p_k - 2)) - (2s + (z_k - 1))(s^2 + (p_k-2)s - 2p_k) = 0$$

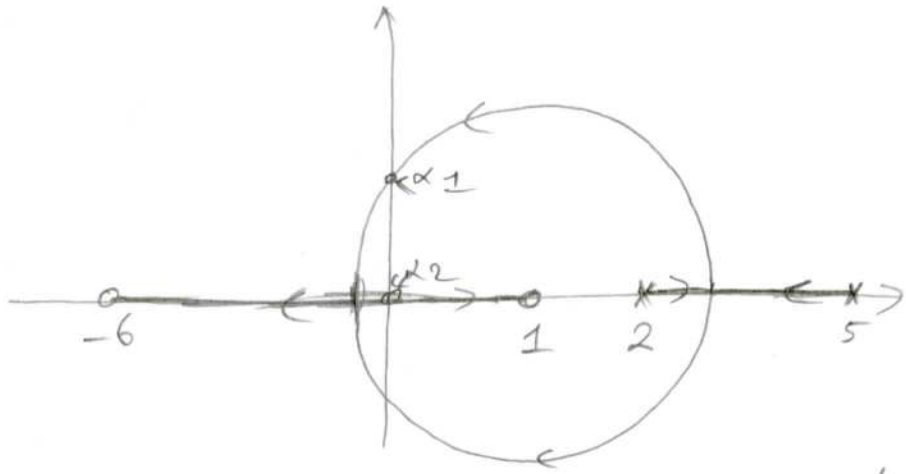
$$\Rightarrow \boxed{[z_k - p_k + 1]s^2 + [4p_k - 2z_k]s + [p_k z_k + 2z_k - 2p_k] = 0} \quad (*)$$

let's then set $p_k = -5$ and $z_k = 6$, then by $(*)$ we get:

$$[6 - 5 + 1]s^2 + [-20 - 12]s + [-30 + 12 + 10] = 0$$

$$\Rightarrow 2s^2 - 32s - 8 = 0 \Rightarrow \begin{cases} P_1 = 2.89 \\ P_2 = -0.2462 \end{cases}$$

Therefore, root locus can be obtained as follows.



In order to ensure the stability, we need to pick $\alpha_1 < \alpha_k < \alpha_2$ to have all the closed loop poles in the LHP.

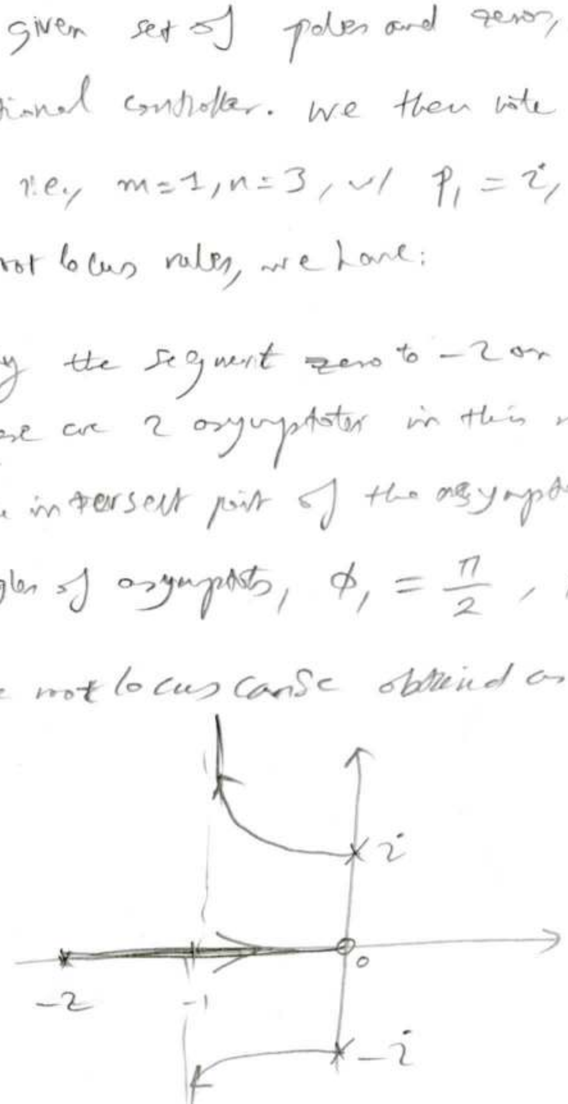
Hence, controller is $K = \alpha_k \dfrac{s+6}{s-5}$ which is not asymptotically stable by its own.

(d)

For the given set of poles and zeros, we shall consider $K = \alpha_k$, i.e., only a proportional controller. We then note that overall we have 3 poles and one zero, i.e., $m=1, n=3$, w/ $p_1 = i$, $p_2 = -i$, $p_3 = -2$ and $z_1 = 0$, therefore, recalling root locus rules, we have:

- only the segment zero to -2 on real axis belongs to root locus
- there are 2 asymptotes in this root locus
- the intersect point of the asymptotes is: $e = \dfrac{i - i - 2 - 0}{3-1} = -1$, w/ angles of asymptotes, $\phi_1 = \frac{\pi}{2}$, $\phi_2 = \frac{\pi + 2\pi}{2} = 3\pi/2$.

thus, the root locus can be obtained as follows:



In fact referring to this root locus, any $\alpha_k > 0$ works and ensures the stability of the closed-loop system. Hence, controller is $K = \alpha_k$ which is not asymptotically stable by its own.

P6.19:

For the given dynamics:

$$\begin{cases} J\dot{\omega} + (b_1+b_2)\omega = \tau + gr(m_1-m_2) \\ v_1 = r\omega \end{cases} \quad (*)$$

we would like to design a controller of the type $\tau = K(\bar{x}_1 - x_1)$, so that $x_1 = \int_0^t v_1(t)\,dt$ tracks the constant reference, $\bar{x}_1$. In order to proceed, we first note that our objective would be achieved if the closed-loop dynamics in terms of $e(t) = \bar{x}_1 - x_1(t)$ would be asymptotically stable because this way, $\lim_{t\to\infty} e(t) = 0 \Rightarrow \lim_{t\to\infty} x_1(t) = \bar{x}_1$, i.e., tracking is achieved. Having said this point, we shall rewrite dynamics $(*)$ in terms of $e(t)$:

$$\left.\begin{array}{l} v_1 = r\omega \Rightarrow \omega = \dfrac{v_1}{r} \xrightarrow[\text{constant}]{r \text{ is a}} J\dfrac{\dot{v}_1}{r} + (b_1+b_2)\dfrac{v_1}{r} = \tau + gr(m_1-m_2) \\ x_1 = \int_0^t v_1(\tau)\,d\tau \Rightarrow \dot{x}_1 = v_1(t) \Rightarrow \dot{v}_1(t) = \ddot{x}_1 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} J\dfrac{\ddot{x}_1}{r} + (b_1+b_2)\dfrac{\dot{x}_1}{r} = \tau + gr(m_1-m_2) \\ \text{Also,} \quad \tau = K(\bar{x}_1 - x_1) = Ke(t) \\ \qquad e = \bar{x}_1 - x_1 \xrightarrow[\text{constant}]{\bar{x}_1 \text{ is}} \begin{cases} \dot{e} = -\dot{x}_1 \\ \ddot{e} = -\ddot{x}_1 \end{cases} \end{array}\right\} \Rightarrow \boxed{\frac{J}{r}(-\ddot{e}) + (b_1+b_2)\left(\frac{-\dot{e}}{r}\right) = Ke + gr(m_1-m_2)} \quad (**)$$

Applying laplace on $(**)$ yields:

$$-s^2 \frac{J}{r} E(s) - \frac{(b_1+b_2)}{r} sE(s) = KE(s) + \frac{gr(m_1-m_2)}{s} \quad (***)$$

where we note Laplace is a linear operator and that m_1, m_2, g, r, b_1, b_2 and J are constant parameters. Grouping terms in $(***)$, we obtain:

$$E(s)\left[\frac{J}{r}s^2 + \frac{(b_1+b_2)}{r}s + K\right] = \frac{gr(m_2-m_1)}{s} \Rightarrow E(s) = \frac{\dfrac{gr(m_2-m_1)}{s}}{\dfrac{J}{r}s^2 + \dfrac{(b_1+b_2)}{r}s + K}$$

$$\Rightarrow \boxed{E(s) = \frac{gr(m_2-m_1)}{\frac{J}{r}s^3 + \frac{(b_1+b_2)}{r}s^2 + Ks}} \quad (\star)$$

In order for $E(s)$ to be asymptotically stable, we need to ensure that the poles of $E(s)$, i.e., the roots of its denominator have negative real part. In order to do so, we use root locus technique:

$$\frac{J}{r} s^3 + \left(\frac{b_1+b_2}{r}\right) s^2 + kS = 0 \Rightarrow k + \frac{(b_1+b_2)}{r} s + \frac{J}{r} s^2 = 0$$

$$\Rightarrow \boxed{1 + \frac{1}{\frac{J}{r} s^2 + \frac{(b_1+b_2)}{r} s} k = 0} \quad (**)$$

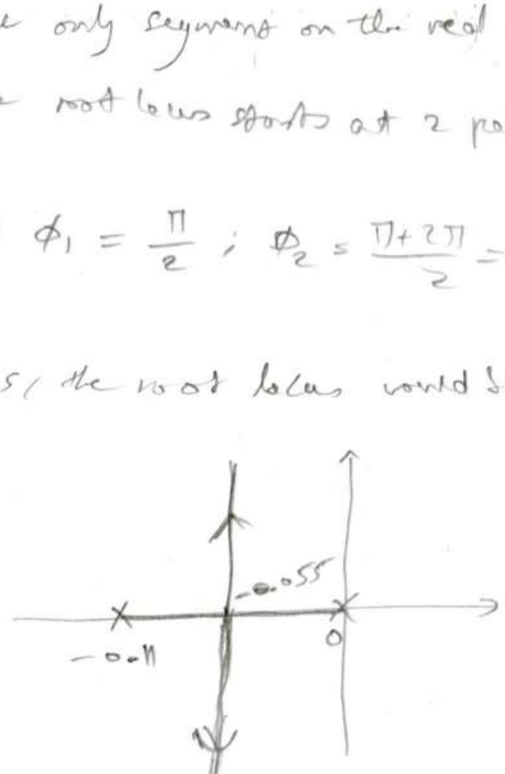
Resorting to (**), $L \equiv \frac{1}{\frac{J}{r} s^2 + \left(\frac{b_1+b_2}{r}\right) s}$ in our problem based on which we get the root locus as follows; where beforehand we plug in the numeric values for parameters: $r = 1m$, $J = J_1 + J_2 + r^2(m_1+m_2) = 2 \times 200 + (1000+800)$
$\Rightarrow J = 2200$

$$\frac{b_1+b_2}{r} = \frac{2 \times 120}{1} = 240 \Rightarrow \boxed{L = \frac{1}{2200 s^2 + 240 s}} \quad (**)$$

- L has 2 poles, $P_1 = 0$, $P_2 = -0.11$ and no zeros i.e., $m=0, n=2$
- the only segment on the real axis that belongs to root locus is $[-0.11, 0]$
- the root locus starts at 2 poles w/ 2 asymptotes intersecting at $c = \frac{0 - 0.11 - 0}{2}$
$\Rightarrow \underline{c = -0.055}$

w/ $\phi_1 = \frac{\pi}{2}$; $\phi_2 = \frac{\pi + 2\pi}{2} = \frac{3\pi}{2}$

Thus, the root locus would be as follows.



According to the root locus, any $k > 0$ would yield poles of $E(s)$ to be on LHP and thus ensures its asymptotic stability. This further ensures asymptotic tracking of $\theta_r(t)$ based on our previous arguments. [illegible]