MAE 143B - Homework 9

7.1

b)

Consider the transfer function

$$G(s) = \frac{s+0.1}{(s+1)(s+10)} = \frac{1}{100} \cdot \frac{1+10s}{(1+s)(1+0.1s)},$$

which has a first-order zero at $z_1 = -0.1$ and first-order poles at $p_1 = -1$ and $p_2 = -10$, respectively, all of them having negative real parts. For small values of ω, we recover the DC gain

$$\tilde{K} = \lim_{\omega \downarrow 0} |G(j\omega)| = \frac{1}{100} = -40\text{dB}.$$

Having this finite limit, our straight-line approximation for the magnitude plot must start with a horizontal line at -40dB. We can then use the poles and zeros to construct our approximation for increasing frequency ω. The lowest frequency is associated with the zero z_1, which will increase both magnitude and phase. The following poles at p_1 and p_2 each decrease magnitude and phase (see the examples in your lecture notes). The missing piece of information is the phase for small frequencies ω, which by positive DC gain has to be $0°$. The resulting Bode plot is in Fig. 1, where the exact plot can be reconstructed in MATLAB using the commands `zpk` (you could also use `tf` or `ss`, whichever you prefer) and `bode`:

```
>> sys = zpk([-0.1],[-1 -10],1)

sys =

    (s+0.1)
  ------------
  (s+1) (s+10)

Continuous-time zero/pole/gain model.

>> bode(sys);
```

e)

We have

$$G(s) = \frac{s+10}{1-0.1s+s^2} = 10 \cdot \frac{1+0.1s}{1-0.1s+s^2},$$

which has a pair of second-order complex poles with $\omega_n = 1$, $\zeta = -0.05$ and a first-order zero at $z_1 = -10$. The DC gain is $\tilde{K} = 10 = 20$dB. Straight-line approximation and exact Bode plot are displayed in Fig. 2. In this figure, the MATLAB phase plot is adjusted by $360°$, which is an equivalent representation (periodicity of sin and cos).

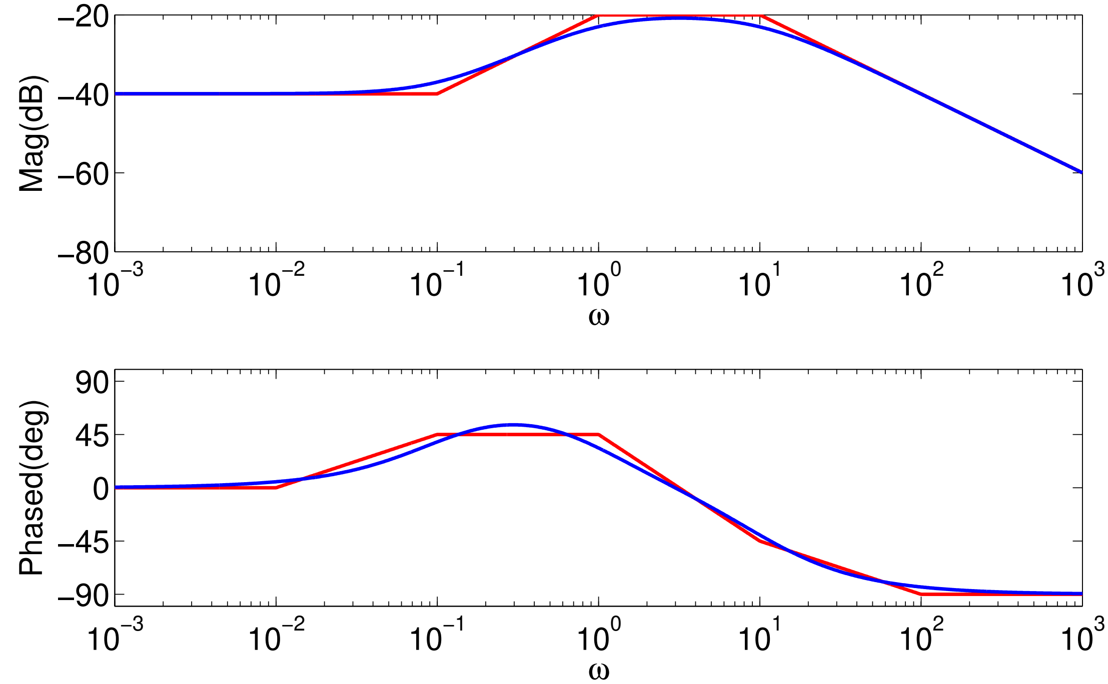

Figure 1: Bode plot for part b), straight-line approximation in red.

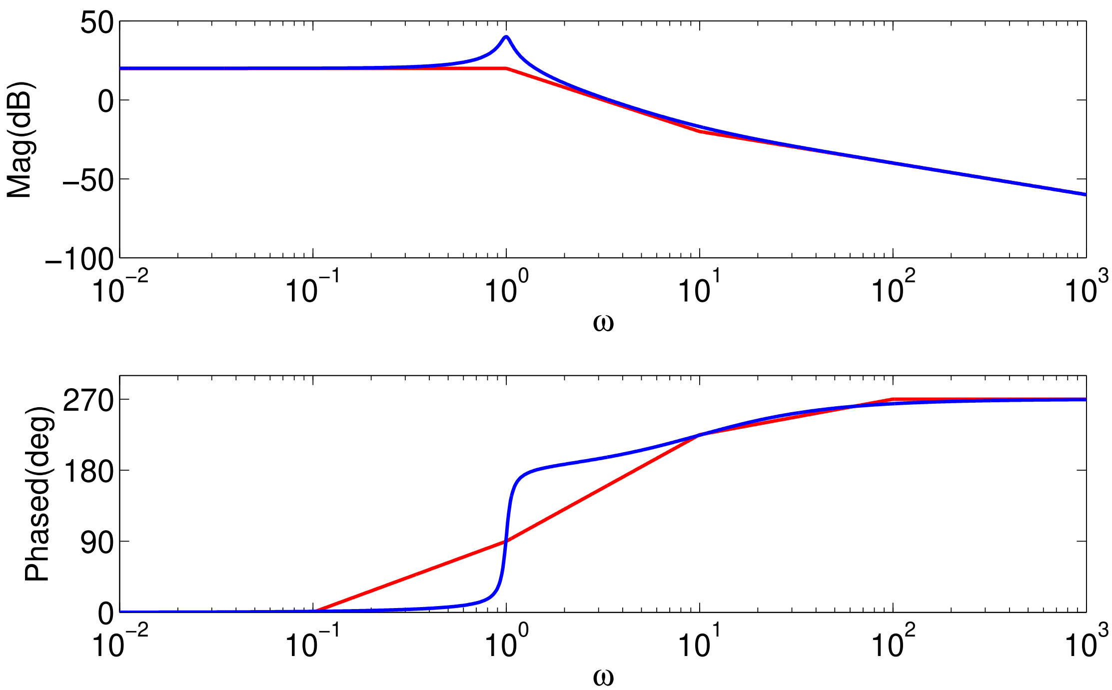

Figure 2: Bode plot for part e), straight-line approximation in red.

h)

In this problem, we have

$$G(s) = \frac{1+s}{s(s+10)} = \frac{1}{10} \cdot \frac{1+s}{s(1+0.1s)},$$

which has simple poles at $p_1 = 0$, $p_2 = -10$ and a simple zero at $z_1 = -1$. The magnitude response has an offset of $\tilde{K} = 0.1 = -20\text{dB}$, such that the asymptote corresponding the pole at zero (integrator) reaches 0dB at $\omega = 0.1$. Moreover, the integrator pole shifts the phase plot by $-\pi/2$. See Fig. 3 for the plots.

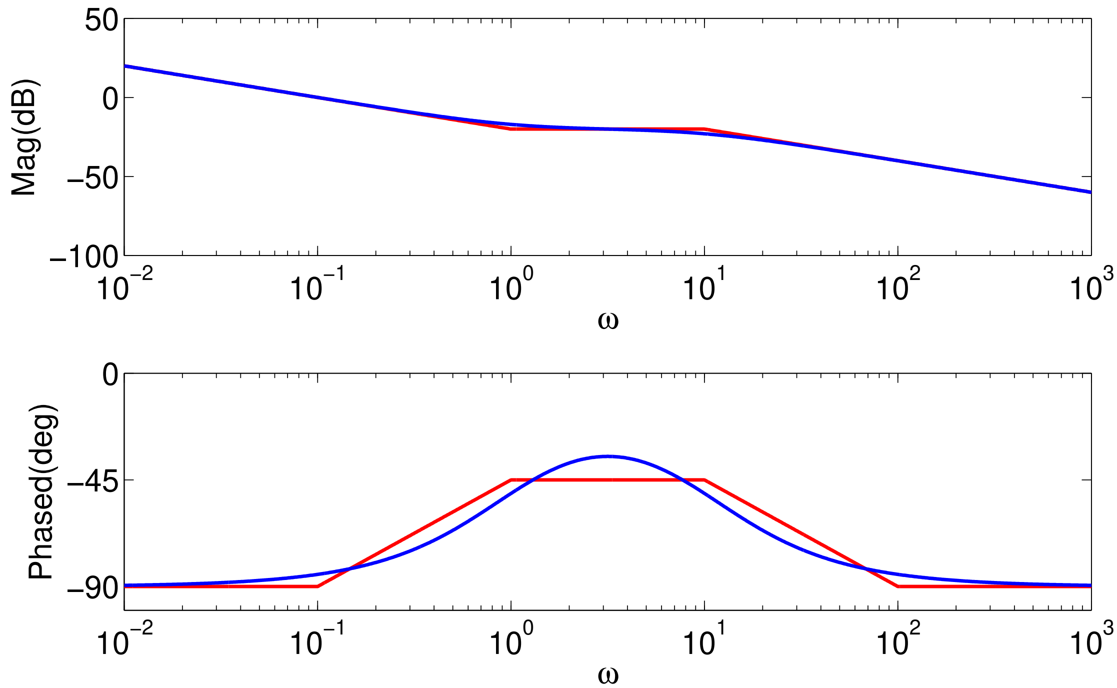

Figure 3: Bode plot for part h), straight-line approximation in red.

7.2

The three plots are shown in Figures 4-6. You can reconstruct these using the MATLAB command `nyquist`.

The first and third polar plots have no encirclements of the point -1, implying stability under negative unit feedback for these two transfer functions by the Nyquist stability criterion. The second polar plot (zoom into yours) has an encirclement of -1. However, the encirclement is in counterclockwise direction. The plot also shows you that the closed-loop system under negative feedback remains stable for gains larger than approximately 1/10. Hence, all systems are stable under negative unit feedback. Feel free to confirm these observations using your root locus skills.

Notice that when doing the third polar plot via MATLAB, there are some scaling issues due to the integrator component of $G(s)$. However, these can be resolved by specifying a desired frequency range for the command `nyquist`.

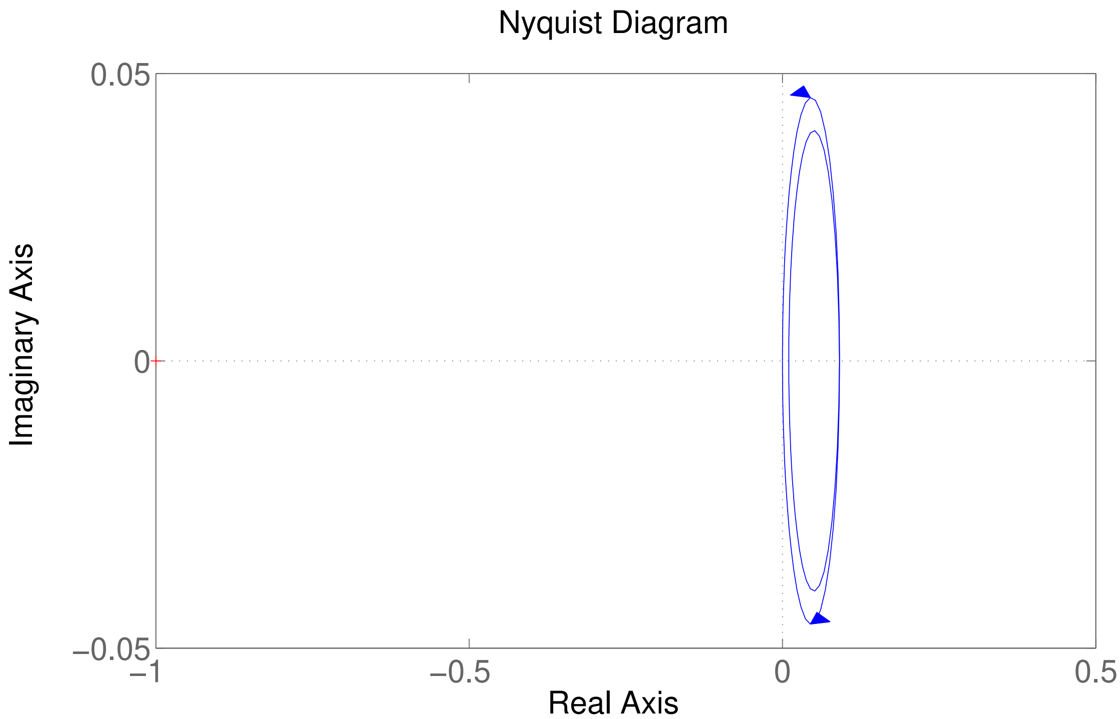

Figure 4: Polar plot for part b).

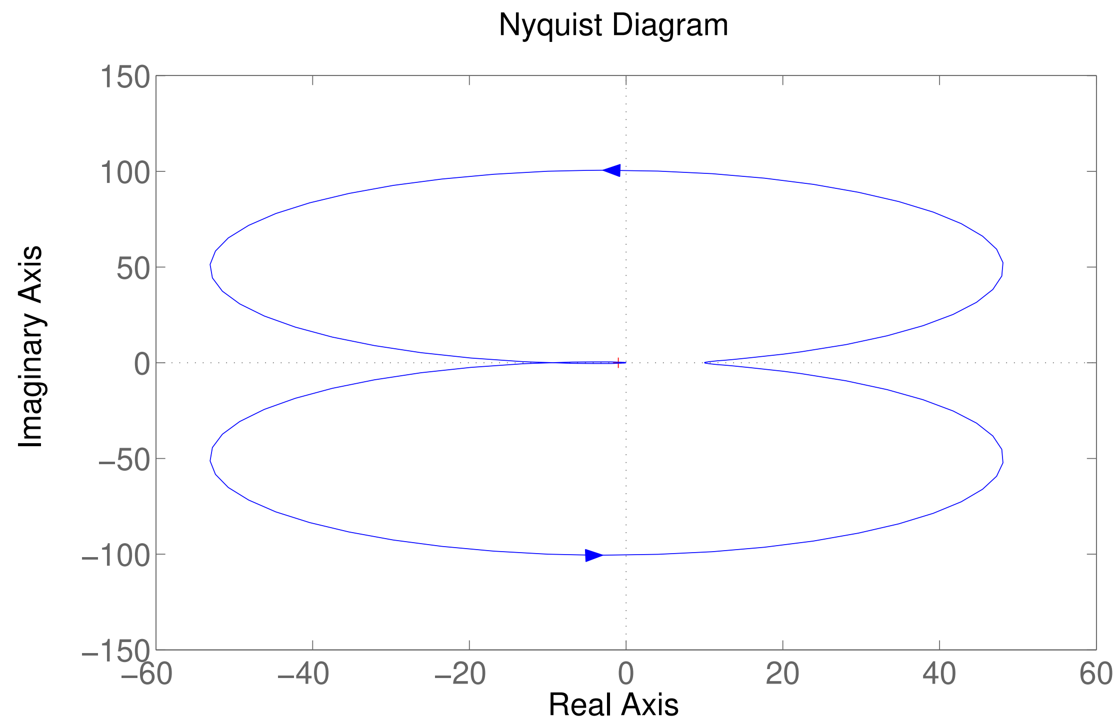

Figure 5: Polar plot for part e).

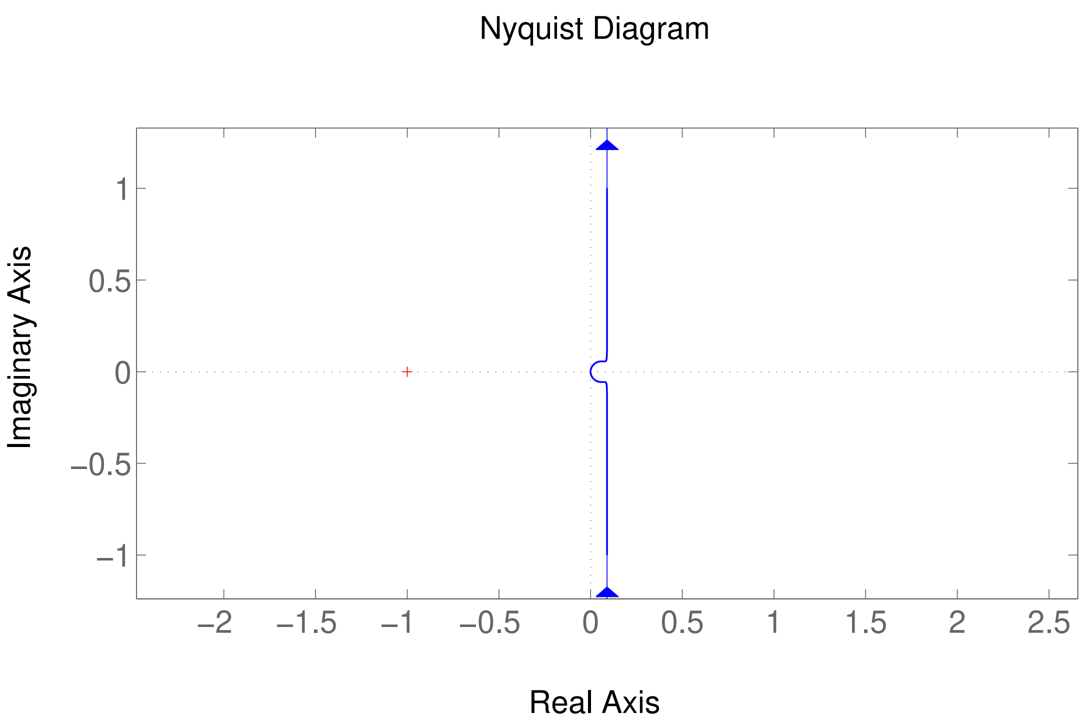

Figure 6: Polar plot for part h).

7.3

In this problem, we seek minimum-phase rational transfer functions to fit the given magnitude plots. For minimum-phase, we have to require that all poles and zeros of our rational transfer functions have strictly negative real parts.

b)

For the first plot, we have a DC gain $\tilde{K} = 20\text{dB} = 10$. The remaining behavior can occur when having a pole at $p_1 = -1$ and a zero at $z_1 = -10$. Combining these, we get the minimum-phase transfer function

$$G(s) = 10 \cdot \frac{1 + 0.1s}{1 + s}.$$

d)

Again, we have DC gain $\tilde{K} = 20\text{dB} = 10$. We can cover the remaining behavior by two poles at $p_1 = -1$ and a zero at $z_1 = -10$, which gives the transfer function

$$G(s) = 10 \cdot \frac{1 + 0.1s}{(1 + s)^2}.$$

h)

Again, we have DC gain $\tilde{K} = 20\text{dB} = 10$. We can cover the remaining behavior by second-order complex poles with $\omega_n = 1$, $\zeta = 0.1$ and a first-order zero at $z_1 = -10$, which gives the transfer function

$$G(s) = 10 \cdot \frac{1 + 0.1s}{1 + 0.2s + s}.$$

To see how the damping factor $\zeta = 0.1$ is reasonable, have a look at Fig. 7.4 in your notes.

7.11

In terms of v_1, the ODE reads

$$\dot{v}_1 + \frac{b_1 + b_2}{J} v_1 = \frac{r}{J}(\tau + gr(m_1 - m_2)) = \frac{r}{J}\tau,$$

where we have used $m_1 = m_2$. You have already seen how to compute the transfer function from τ to v_1 in a previous homework, which resulted in

$$G_1(s) = \frac{r}{sJ + b_1 + b_2}.$$

By $X(s) = V_1(s)/s$, we then have

$$G(s) = \frac{1}{s} G_1(s) = \frac{r}{Js^2 + (b_1 + b_2)s}.$$

To analyze the tracking behavior, take a look at the sensitivity transfer function

$$S(s) = \frac{1}{1 + G(s)K(s)}.$$

Now note that this transfer function represents $G(s)K(s)$ with negative unit feedback. Hence, given a dynamic control law $K(s)$, we can use the Nyquist stability criterion to determine whether or not asymptotic tracking is possible. That is, we must guarantee asymptotic stability of the sensitivity function under negative unit feedback for asymptotic tracking of a constant reference input to be possible. Provided asymptotic stability of S, Lemma 4.1 in the lecture notes guarantees asymptotic tracking of a constant reference if $G(s)K(s)$ has a pole at $s = 0$. However, $G(s)$ already has a pole at the origin, implying that we do not require an integrator component in $K(s)$.

Consider that we already have an integrator component in $G(s)$, it may seem reasonable to first attempt using a static gain $K(s) = \hat{K}$ for our controller. We then get

$$G(s)K(s) = \frac{r}{Js^2 + (b_1 + b_2)s}\hat{K} = \frac{r\hat{K}}{b_1 + b_2} \cdot \frac{1}{s(1 + sJ/(b_1 + b_2))}.$$

Given this transfer function, the value used for $\hat{K}$ determines the vertical shift of our Bode magnitude plot. Given the problem data, the second pole of our transfer function is at

$$p_2 = -\frac{b_1 + b_2}{J} = 0.11/\text{second}.$$

Figure 7 shows the resulting Bode plot for $\hat{K} = 1$ (negative unit feedback). As we can see, the phase asymptotically approaches $-180°$ from above, implying infinite gain margin. Crossing $0dB$ at $\omega < 10^{-2}$, we have a phase margin of approximately $90°$ (more precise: $87.6°$). Again, because the phase asymptotically approaches $-180°$ from above, we have positive phase margin for any $\hat{K} > 0$. A Nyquist plot for $\hat{K} = 1$ is displayed in Fig. 8 and confirms our observations. That is, the point $s = -1$ is encircled through ∞, but in counterclockwise direction. Moreover, we can see that any positive gain (effectively moving our critical point along the negative real axis) results in asymptotic stability of $S(s)$ and thus asymptotic tracking.

In summary, we can chose any constant gain $\hat{K} > 0$ for our control law to achieve asymptotic tracking of an arbitrary constant reference input $\bar{x}_1$.

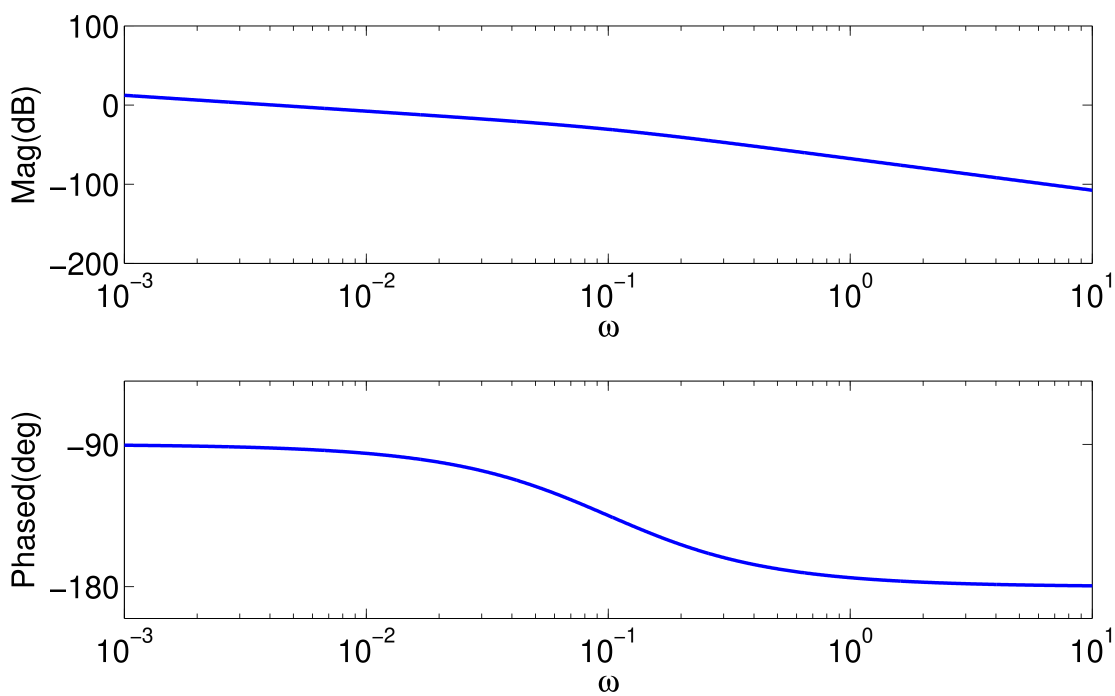

Figure 7: Bode plot for P7.11 with $\hat{K} = 1$.

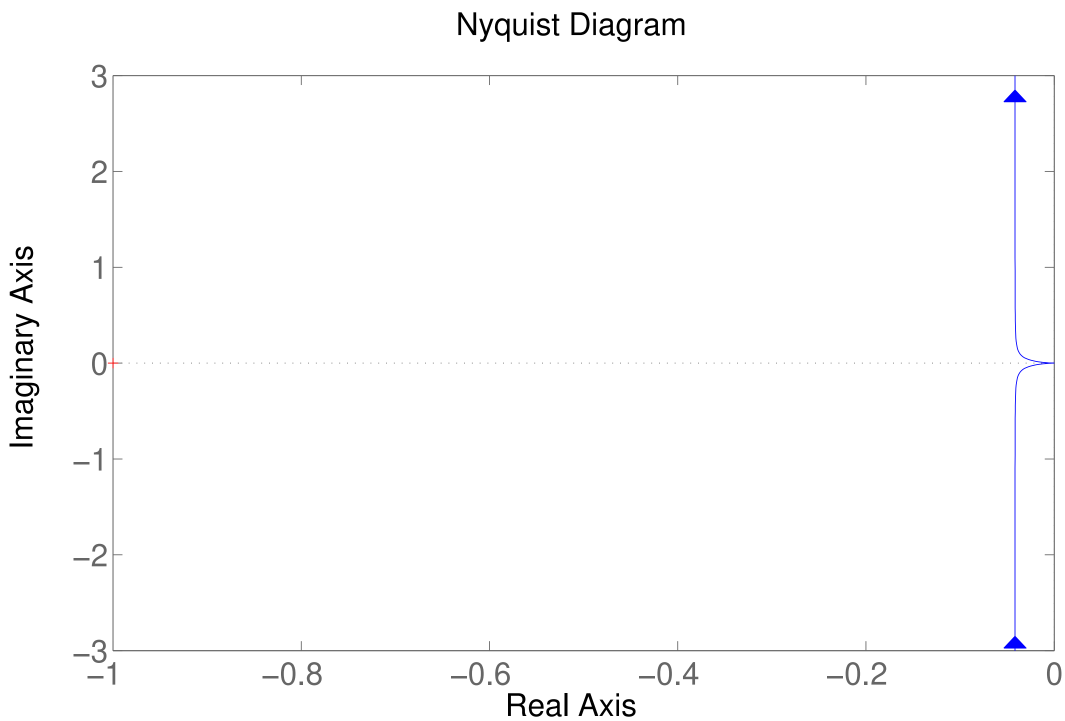

Figure 8: Nyquist plot for P7.11 with $\hat{K} = 1$.