MAE143 B - Linear Control - Spring 2014
Midterm, May 1st

Instructions:

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities
2. You have 70 minutes
3. Most questions show you the answer, and you're asked to prove it. Use the given answers to move on to the next question. Do not get stuck!
4. Write your answers on your "Blue Book"
5. Do not forget to write your name, student number, and instructor

Questions:

1. **System Modelling.**

   The second-order ordinary differential equation:

   $$m\,\ddot{x} + b\,\dot{x} = m\,g,$$

   where x is the object's position, describes the motion of an object of mass m dropping vertically under constant gravitational acceleration, g, and air resistance, $-b\,\dot{x}$.

   (a) (3 points) Ignoring initial conditions, use the Laplace transform to show that

   $$X(s) = G(s)\,U(s), \qquad G(s) = \frac{1}{s\left(s + \frac{b}{m}\right)}, \qquad U(s) = \frac{g}{s}$$

   (b) (2 points) Use the inverse Laplace transform to show that:

   $$u(t) = g, \qquad x(t) = \frac{m\,g}{b}t - \frac{m^2}{b^2}\left(1 - e^{-\frac{b}{m}t}\right)g, \qquad t \geq 0.$$

   (c) (2 points (bonus)) Show that the complete solution is:

   $$x(t) = x_0 + \frac{m}{b}\left(\dot{x}_0 - \frac{m\,g}{b}\right)\left(1 - e^{-\frac{b}{m}t}\right) + \frac{m\,g}{b}t, \qquad t \geq 0,$$

   where x_0 and $\dot{x}_0$ are the initial conditions at $t = 0$.

2. **Linear Estimation.**

In this question you will design a *linear estimator*:

$$\hat{x} = Hz, \qquad\qquad z = x + v,$$

to estimate the position of the falling object modeled in Q1. The signal v is *noise* and your task is to design the estimator transfer-function H to achieve the best possible estimate $\hat{x}$ in the presence of noise and unknown initial conditions.

(a) (2 points) Draw a block-diagram relating the signals $\hat{x}$, x, u and v. Is there feedback in your diagram? Do you need to know the initial conditions to implement this estimator? What about the parameters m, b and g?

(b) (2 points) Show that the *estimation error* is

$$e = x - \hat{x} = SGu - Hv, \qquad\qquad S = 1 - H.$$

(c) (1 point) When $v = 0$ what is the choice of the estimator transfer-function H that makes the estimation error equal to zero, i.e. $e = 0$? Explain your answer.

(d) (1 point) Explain why it is not possible to make $e = 0$ when $v \neq 0$.

(e) (3 points) Let $v = 0$ and u be the input you calculated in Q1(b). Show that if S is asymptotically stable and has two zeros at zero then the estimation error converges asymptotically to zero, that is:

$$\lim_{t \to \infty} e = 0$$

if the initial conditions are zero.

(f) (3 points) Set $b = m = 1$ and calculate the transfer-functions S and SG when H is the *low-pass filter*:

$$H(s) = \frac{a^2 + 2as}{(s+a)^2}, \qquad a > 0.$$

Does this estimator satisfy the conditions in part (e)?

(g) (2 points (bonus)) Show that the conditions in part (e) are also enough for the estimation error to converge to zero even if the initial conditions are not zero.

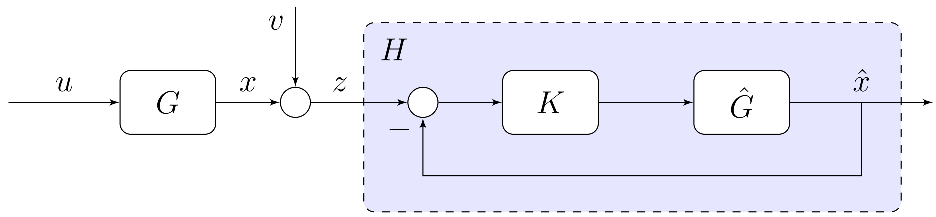

Figure 1: Figure for Q3

3. **Observer.** In this question you will design a special type of linear estimator using feedback techniques. Instead of computing the estimator transfer-function H directly, you will compute the feedback "controller" K shown in Fig. 1. When $\hat{G}$ is a replica of the system tranfer-function G the resulting estimator is known as an *observer*.

   (a) (2 points) Show that the transfer function from z to $\hat{x}$ in Fig. 1 is:

   $$H = \frac{K\hat{G}}{1 + K\hat{G}}, \qquad S = 1 - H = \frac{1}{1 + K\hat{G}}.$$

   (b) (2 points) What are the poles and zeros of S and SG when $\hat{G} = G$?

   (c) (3 points) Let $v = 0$, $\hat{G} = G$, and u be the input you calculated in Q1(b). Show that if S is asymptotically stable and K has one pole at zero then the estimation error converges asymptotically to zero, that is:

   $$\lim_{t \to \infty} e = 0.$$

   Hint: Use Q2(e).

   (d) (3 points) Calculate the estimator transfer-function H and the transfer-functions S and SG when

   $$K(s) = \frac{(K_p + K_d s)}{s}\left(s + \frac{b}{m}\right), \qquad \hat{G}(s) = G(s).$$

   (e) (3 points) Show that the observer in part (d) with the particular choice of gains:

   $$K_d^2 = 4K_p, \qquad K_d > 0$$

   achieves asymptotic convergence of the estimation error to zero.

   (f) (2 points (bonus)) What is the difference between the estimator in Q2(f) and Q3(e)?