MAE143 B - Linear Control - Spring 2014
Midterm, May 1st

Instructions:

1. This exam is open book. You may use whatever written materials you choose, including your class notes and textbook. You may use a hand calculator with no communication capabilities
2. You have 70 minutes
3. Most questions show you the answer, and you're asked to prove it. Use the given answers to move on to the next question. Do not get stuck!
4. Write your answers on your "Blue Book"
5. Do not forget to write your name, student number, and instructor

Questions:

1. **System Modelling.**

The second-order ordinary differential equation:

$$m\,\ddot{x} + b\,\dot{x} = m\,g,$$

where x is the object's position, describes the motion of an object of mass m dropping vertically under constant gravitational acceleration, g, and air resistance, $-b\,\dot{x}$.

(a) (3 points) Ignoring initial conditions, use the Laplace transform to show that

$$X(s) = G(s)\,U(s), \qquad G(s) = \frac{1}{s\left(s + \frac{b}{m}\right)}, \qquad U(s) = \frac{g}{s}$$

Solution: Dividing by $m > 0$ and ignoring initial conditions:

$$\mathcal{L}\left\{\ddot{x} + \frac{b}{m}\dot{x}\right\} = s^2 X(s) + \frac{b}{m} s X(s) = s\left(s + \frac{b}{m}\right) X(s) = \frac{g}{s} = \mathcal{L}\{g\} \quad \text{(+2 points)}$$

from which

$$X(s) = \frac{1}{s\left(s + \frac{b}{m}\right)} U(s), \qquad U(s) = \frac{g}{s}. \qquad \text{(+1 point)}$$

(b) (2 points) Use the inverse Laplace transform to show that:

$$u(t) = g, \qquad x(t) = \frac{m\,g}{b} t - \frac{m^2}{b^2}\left(1 - e^{-\frac{b}{m}t}\right) g, \qquad t \geq 0.$$

Solution: Calculating

$$\begin{aligned} X(s) &= \frac{1}{s\left(s+\frac{b}{m}\right)}\frac{g}{s} \\ &= \frac{g}{s^2\left(s+\frac{b}{m}\right)} \end{aligned}$$

and applying the Laplace inverse

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s\left(s+\frac{b}{m}\right)}\frac{g}{s}\right\} && \text{(+1 point)} \\ &= k_1 t + k_2 + k_3 e^{-\frac{b}{m}t}, \quad t \geq 0. \end{aligned}$$

Using residues to calculate the k's:

$$\begin{aligned} k_1 &= \lim_{s\to 0} s^2 X(s) = \frac{m\,g}{b}, && \text{(+0.5 point)} \\ k_2 &= \lim_{s\to 0}(d/ds)s^2 X(s) = \lim_{s\to 0}\frac{-g}{\left(s+\frac{b}{m}\right)^2} = -\frac{m^2\,g}{b^2}, \\ k_3 &= \lim_{s\to -\frac{b}{m}}\left(s+\frac{b}{m}\right)X(s) = \frac{m^2\,g}{b^2}. \end{aligned}$$

Also

$$u(t) = \mathcal{L}^{-1}\left\{\frac{g}{s}\right\} = g, \quad t \geq 0. \qquad \text{(+0.5 point)}$$

(c) (2 points (bonus)) Show that the complete solution is:

$$x(t) = x_0 + \frac{m}{b}\left(\dot{x}_0 - \frac{m\,g}{b}\right)\left(1 - e^{-\frac{b}{m}t}\right) + \frac{m\,g}{b}t, \qquad t \geq 0,$$

where x_0 and $\dot{x}_0$ are the initial conditions at $t = 0$.

Solution:
We set the input, g, to zero:

$$\mathcal{L}\left\{\ddot{x} + \frac{b}{m}\dot{x}\right\} = s^2 X(s) - s x_0 - \dot{x}_0 + \frac{b}{m}\left(sX(s) - x_0\right) = 0.$$

Rearranging:

$$\begin{aligned} X(s) &= G(s)\left(\left(s+\frac{b}{m}\right)x_0 + \dot{x}_0\right) \\ &= \frac{x_0}{s} + \frac{\dot{x}_0}{s\left(s+\frac{b}{m}\right)}. \end{aligned}$$

Using the inverse Laplace transform:

$$
\begin{aligned}
x(t) &= \mathcal{L}^{-1}\left\{\frac{x_0}{s} + \frac{\dot{x}_0}{s\left(s+\frac{b}{m}\right)}\right\} \\
&= k_1 + k_2 e^{-\frac{b}{m}t}.
\end{aligned}
$$

Using residues to calculate the k's:

$$
\begin{aligned}
k_1 &= \lim_{s\to 0} sX(s) = x_0 + \frac{m}{b}\dot{x}_0, \\
k_2 &= \lim_{s\to -\frac{b}{m}} \left(s+\frac{b}{m}\right) X(s) = -\frac{m}{b}\dot{x}_0
\end{aligned}
$$

from which

$$
x(t) = x_0 + \frac{m}{b}\dot{x}_0\left(1 - e^{-\frac{b}{m}t}\right)
$$

The complete solution is the sum of these two. (+2 points)

2. **Linear Estimation.**

In this question you will design a *linear estimator*:

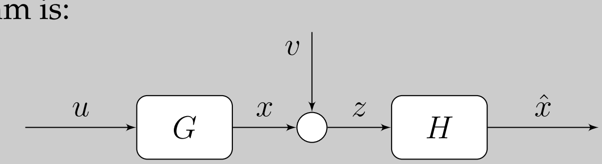
$$
\hat{x} = Hz, \qquad\qquad z = x + v,
$$

to estimate the position of the falling object modeled in Q1. The signal v is *noise* and your task is to design the estimator transfer-function H to achieve the best possible estimate $\hat{x}$ in the presence of noise and unknown initial conditions.

(a) (2 points) Draw a block-diagram relating the signals $\hat{x}$, x, u and v. Is there feedback in your diagram? Do you need to know the initial conditions to implement this estimator? What about the parameters m, b and g?

Solution:

A possible diagram is:



(+1 point)

By the way, it is already printed on the next page in the original exam :).

There is no feedback involved. The estimator makes use of the measurement z and needs no information on initial condition or parameters of G. (+1 point)

(b) (2 points) Show that the *estimation error* is

$$
e = x - \hat{x} = SGu - Hv, \qquad\qquad S = 1 - H.
$$

Solution: From the diagram:

$$\hat{x} = Hz = H(x+v) = HGu + Hv \qquad \text{(+1 point)}$$

and

$$e = x - \hat{x} = Gu - HGu - Hv = (1-H)Gu - v = SGu - v. \qquad \text{(+1 point)}$$

(c) (1 point) When $v = 0$ what is the choice of the estimator transfer-function H that makes the estimation error equal to zero, i.e. $e = 0$? Explain your answer.

Solution: When $v = 0$

$$e = (1-H)Gu \qquad \text{(+0.5 point)}$$

and the choice $H = 1$ will make $e = 0$ no matter G.

This means that if there is no noise, v, them your best estimate is the exact measurement x. (+0.5 point)

(d) (1 point) Explain why it is not possible to make $e = 0$ when $v \neq 0$.

Solution: When $v \neq 0$

$$e = (1-H)Gu + Hv \qquad \text{(+0.5 point)}$$

The choice $H = 1$ makes $e = v$. On the other hand, $H = 0$, makes $e = Gu = x$ which is not zero! There is no choice that can simultaneously zero e. (+0.5 point)

This argument is also true when G and H are dynamic since this relationship holds now under the light of the frequency response where:

$$(1 - H(j\omega))G(j\omega) \quad \text{and} \quad H(j\omega)$$

cannot be simultaneously zero if $G(j\omega) \neq 0$.

(e) (3 points) Let $v = 0$ and u be the input you calculated in Q1(b). Show that if S is asymptotically stable and has two zeros at zero then the estimation error converges asymptotically to zero, that is:

$$\lim_{t\to\infty} e = 0$$

if the initial conditions are zero.

Solution: If S is asymptotically stable and has two zeros at zero

$$S(s) = s^2 F(s), \qquad \text{(+1 point)}$$

where $F(s)$ is asymptotically stable and does not have any zeros at zero, that is $F(0) \neq 0$. Therefore:

$$\begin{aligned} T(s) &= S(s)G(s) && \text{(+1 point)} \\ &= s^2 F(s) \frac{1}{s\left(s+\frac{b}{m}\right)} \\ &= \frac{sF(s)}{s+\frac{b}{m}} \end{aligned}$$

is also asymptotically stable because $b > 0$, $m > 0$. In response to a constant input $u(t) = g$, $t \geq 0$, the steady-state estimation error is another constant:

$$e_{ss}(t) = T(0)g = 0,$$

and $\lim_{t\to 0} e(t) = e_{ss}(t) = 0$. (+1 point)

(f) (3 points) Set $b = m = 1$ and calculate the transfer-functions S and SG when H is the *low-pass filter*:

$$H(s) = \frac{a^2 + 2as}{(s+a)^2}, \qquad a > 0.$$

Does this estimator satisfy the conditions in part (e)?

Solution: Calculate:

$$\begin{aligned} S &= 1 - H, && \text{(+1 point)} \\ &= 1 - \frac{a^2 + 2as}{s^2 + 2as + a^2}, \\ &= \frac{s^2 + 2as + a^2 - a^2 - 2as}{s^2 + 2as + a^2}, \\ &= \frac{s^2}{s^2 + 2as + a^2}, \end{aligned}$$

and

$$\begin{aligned} SG &= \frac{s^2}{s^2 + 2as + a^2} \times \frac{1}{s\left(s+\frac{b}{m}\right)}, && \text{(+1 point)} \\ &= \frac{s}{\left(s^2 + 2as + a^2\right)\left(s+\frac{b}{m}\right)}. \end{aligned}$$

The transfer-function S has two zeros at zero and is asymptotically stable, satisfying conditions in part (e). (+1 point)

(g) (2 points (bonus)) Show that the conditions in part (e) are also enough for the estimation error to converge to zero even if the initial conditions are not zero.

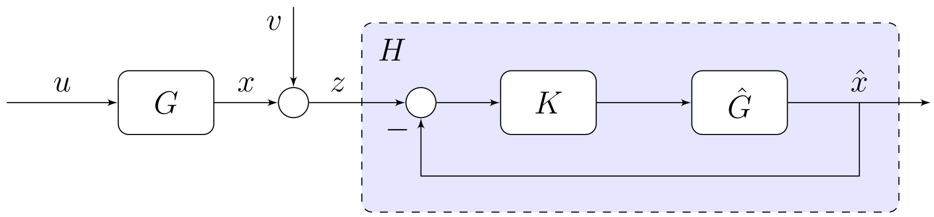


Figure 1: Figure for Q3

Solution: From the solution to Q1(c), the response to the initial condition is

$$X(s) = G(s)U(s), \qquad U(s) = \left(s + \frac{b}{m}\right) x_0 + \dot{x}_0$$

from which

$$e = Su$$

will converge to zero since $U(0) \neq 0$ and $S(0) = 0$. Note that $u(t) = \mathcal{L}^{-1}U(s)$ is impulsive and is zero for any t other than 0.

3. **Observer.** In this question you will design a special type of linear estimator using feedback techniques. Instead of computing the estimator transfer-function H directly, you will compute the feedback "controller" K shown in Fig. 1. When $\hat{G}$ is a replica of the system tranfer-function G the resulting estimator is known as an *observer*.

   (a) (2 points) Show that the transfer function from z to $\hat{x}$ in Fig. 1 is:

$$H = \frac{K\hat{G}}{1 + K\hat{G}}, \qquad S = 1 - H = \frac{1}{1 + K\hat{G}}.$$

Solution: The diagram from z to $\hat{x}$ is a standard feedback diagram from which

$$H = \frac{K\hat{G}}{1 + K\hat{G}}, \qquad \text{(+1 point)}$$

and

$$S = 1 - H = 1 - \frac{K\hat{G}}{1 + K\hat{G}} = \frac{1}{1 + K\hat{G}}. \qquad \text{(+1 point)}$$

   (b) (2 points) What are the poles and zeros of S and SG when $\hat{G} = G$?

Solution: As in a standard feedback connection, the poles of S are the zeros of the characteristic equation $1 + K\hat{G} = 1 + KG = 0$ and the zeros of S are the poles of KG. (+1 point)

The poles of SG are the same as the poles of S because the poles of G cancel the zeros of S. The remaining zeros of SG are therefore poles of K. (+1 point)

   (c) (3 points) Let $v = 0$, $\hat{G} = G$, and u be the input you calculated in Q1(b). Show that if S is asymptotically stable and K has one pole at zero then the estimation error converges asymptotically to zero, that is:

$$\lim_{t \to \infty} e = 0.$$

Hint: Use Q2(e).

Solution:

If K has one pole at zero $K\hat{G} = KG$ has two poles at zero because G has a pole at zero and no zeros. (+1 point)

Therefore, from part (b), S will have two poles at zero. (+1 point)

If it is also asymptotically stable then convergence of the estimation error follows from Q2(e). (+1 point)

   (d) (3 points) Calculate the estimator transfer-function H and the transfer-functions S and SG when

$$K(s) = \frac{(K_p + K_d s)}{s}\left(s + \frac{b}{m}\right), \qquad \hat{G}(s) = G(s).$$

Solution: Calculating:

$$
\begin{aligned}
H &= \frac{\frac{K_p+K_d s}{s}\left(s+\frac{b}{m}\right)\times\frac{1}{s\left(s+\frac{b}{m}\right)}}{1+\frac{K_p+K_d s}{s}\left(s+\frac{b}{m}\right)\times\frac{1}{s\left(s+\frac{b}{m}\right)}},\\
&= \frac{\frac{K_p+K_d s}{s^2}}{1+\frac{K_p+K_d s}{s^2}},\\
&= \frac{K_p+K_d s}{s^2+K_d s+K_p}, && \text{(+1 point)}
\end{aligned}
$$

then

$$
\begin{aligned}
S &= 1-H,\\
&= 1-\frac{K_p+K_d s}{s^2+K_d s+K_p},\\
&= \frac{s^2}{s^2+K_d s+K_p}, && \text{(+1 point)}
\end{aligned}
$$

and finally

$$
\begin{aligned}
SG &= \frac{s^2}{s^2+K_d s+K_p}\frac{1}{s\left(s+\frac{b}{m}\right)},\\
&= \frac{s}{\left(s^2+K_d s+K_p\right)\left(s+\frac{b}{m}\right)}. && \text{(+1 point)}
\end{aligned}
$$

Note the convenient pole-zero cancelations.

(e) (3 points) Show that the observer in part (d) with the particular choice of gains:

$$K_d^2 = 4K_p, \qquad\qquad K_d > 0$$

achieves asymptotic convergence of the estimation error to zero.

Solution: With

$$K_p = a^2, \qquad K_d = 2\sqrt{K_p} = 2a, \qquad a > 0 \qquad \text{(+1 point)}$$

we calculate:

$$S = \frac{s^2}{s^2+K_d s+K_p} = \frac{s^2}{s^2+2as+a^2} = \frac{s^2}{(s+a)^2} \qquad \text{(+1 point)}$$

which is asymptotically stable, two poles at $s=-a<0$, and has two zeros at zero, Q2(e). (+1 point)

(f) (2 points (bonus)) What is the difference between the estimator in Q2(f) and Q3(e)?

Solution:

The estimator in Q3(e) can produce the exact same response as Q2(f) with the choice of parameters from part (e). However, for other choices of K_p it will produce a different response, but always guaranteeing that the estimation error is asymptotically stable when $v = 0$. The closed-loop diagram offers an parametrization of the estimators that always ensure asymptotic stability of the estimator error even if G itself is not asymptotically stable. (+2 points)