MAE143 B - Linear Control - Spring 2015
Final, June 8th

Instructions:

1. This exam is open book. You may use whatever written materials you choose, including your notes and the reader. You may use a calculator with no communication capabilities
2. You have 170 minutes
3. Do not get stuck! Move on and come back later if you have time
4. Write answers on your "Blue Book"
5. Do not forget to write your name, student number, and instructor
6. Initial conditions are zero unless otherwise noted
7. The exam has 7 question with for a total of 62 points and 9 bonus points

Questions:

1. **The Device.**

   The diagram in Fig. 1 represents a tracking device that uses two sensors separated by a distance $\ell > 0$ at positions x and y to locate a target at position z. The device is stationary but can rotate about its center of mass, the point o. The equation:

   $$J\,\ddot{\psi} + b\,\dot{\psi} = u, \tag{1}$$

   where u is a torque applied by a motor connected to the device's center of mass, J is the device's moment of inertia, and b is the device's viscous damping coefficient, is a simplified model for the dynamics of the tracking device.

   (a) (3 points) Draw a block-diagram relating the signals ψ, $\dot{\psi}$, and u without using differentiators.

   (b) (3 points) Calculate the transfer-function from u to ψ.

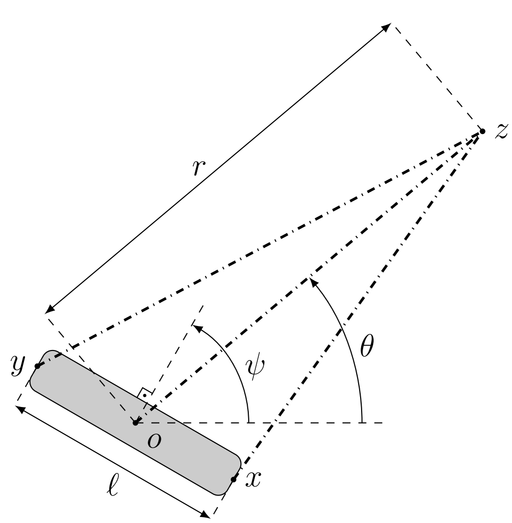

Figure 1: Locating a target z with sensors at x and y

2. **The Sensors.**

   Using the law of cosines, the distance from the sensors to the target can be shown to be:

$$r_x(\psi,\theta,r) = \sqrt{r^2 + (\ell/2)^2 - r\,\ell\sin(\psi-\theta)}, \quad r_y(\psi,\theta,r) = \sqrt{r^2 + (\ell/2)^2 + r\,\ell\sin(\psi-\theta)},$$

   where θ is the angle and r is the distance to the target measured from o with respect to an inertial frame as shown in Fig. 1.

   Many types of sensors can produce signals that are proportional to the distance to the target, for example, sonar, radar or even imaging, depending on the application. Identical sensors at points x and y produce the signals:

$$d_x = \gamma\, r_x, \qquad\qquad d_y = \gamma\, r_y,$$

   where γ is the sensor gain.

   (a) (3 points) Assume that the tracking device is modeled by (1). Draw a block-diagram relating the signals d_x, d_y, ψ, $\dot{\psi}$, r, θ, and u. Use a single block to produce r_x from ψ, θ and r. Do the same for r_y.

3. **Control Part I.**

   One approach to tracking is to use the control law:

$$u = -K\frac{d_y^2 - d_x^2}{2\,\gamma\,\ell} = -K\,\gamma\, r\sin(\psi-\theta). \tag{2}$$

   (a) (3 points) Show that the connection of the tracking device with the control law (2) can be modeled by the nonlinear dynamic equations:

$$J\,\ddot{e} + b\,\dot{e} + K\gamma\, r\sin(e) = -J\,\ddot{\theta} - b\,\dot{\theta}, \qquad\qquad e = \psi - \theta. \tag{3}$$

   (b) (2 points (bonus)) Show that

$$\begin{pmatrix}\dot{x}_1 \\ \dot{x}_2\end{pmatrix} = \begin{pmatrix} x_2 - \dfrac{b}{J}x_1 \\ -\dfrac{K\gamma\, r}{J}\sin(x_1-\theta)\end{pmatrix}, \qquad\qquad e = x_1 - \theta$$

   is a state-space realization for the differential equation (3). *Hint: differentiate 'e' twice.*

   (c) (3 points) Draw the block-diagram associated with the state-space realization in part (b) without using differentiators.

   (d) (2 points) Assume that $\theta = \bar{\theta}$ and $r = \bar{r}$ are constant and show that all possible equilibrium points for the closed-loop tracking system are such that $e = \bar{e} = \kappa\pi$, $\kappa \in \mathbb{Z}$. Interpret the result.

   (e) (2 points (bonus)) Linearize about the equilibrium points to show that the transfer-function of the linearized system between $\tilde{\theta} = \theta - \bar{\theta}$ and $\tilde{e} = e - \bar{e}$ is:

$$S(s) = \frac{\tilde{E}(s)}{\tilde{\Theta}(s)} = -\frac{s^2 + (b/J)s}{s^2 + (b/J)s + \delta K\,\gamma\,\bar{r}/J},$$

   where $\delta = 1$ when κ is even and $\delta = -1$ when κ is odd.

(f) (2 points) Assume that all constants are positive and classify the corresponding equilibria as asymptotically stable or unstable.

(g) (2 points) Can the closed-loop system asymptotically track a target that has a constant angle $\theta(t) = \underline{\theta} \neq \bar{\theta}$, $t \geq 0$? Explain.

(h) (1 point (bonus)) What is the impact of a variable r on the performance of the closed-loop tracking system?

(i) (1 point (bonus)) One way to mitigate the dependence on r is to use the control law:

$$u = -K \frac{d_y - d_x}{\gamma\, \ell}. \tag{4}$$

Show that $u \approx -K(\psi - \theta)$ when $\psi - \theta$ is small and $r \gg \ell$.

4. **Control Part II.**

Consider the more sophisticated control law:

$$u = -Ke, \qquad e = \arcsin\left(\frac{d_y^2 - d_x^2}{\gamma\, \ell \sqrt{2} \sqrt{d_y^2 + d_x^2 - \gamma^2 \ell^2 / 2}}\right). \tag{5}$$

(a) (1 point (bonus)) Show that an exact implementation of the control law (5) implies:

$$u = -Ke, \qquad e = \psi - \theta. \tag{6}$$

(b) (1 point) Draw a block-diagram representing the connection of the linear control law (6) with the linear model (1).

(c) (2 points) Can the closed-loop system asymptotically track a target that has a constant angle $\theta(t) = \bar{\theta}$, $t \geq 0$? Explain.

(d) (2 points) Explain why the closed-loop system cannot track a target that has an angle that depends affinely on time, that is $\theta(t) = \alpha + \beta\, t$, $t \geq 0$.

(e) (2 points) What are the minimum requirements for a controller that uses the signal $e = \psi - \theta$ to asymptotically track a target $\theta(t) = \alpha + \beta\, t$, $t \geq 0$?

You do not need to design a controller. You will do that in the next sections!

Let $L(s)$ be the transfer-function from the input u to ψ calculated in 1(b).
For the rest of the exam consider the data:

$$J = 1, \qquad b = 1$$

In the next questions you will design linear controllers that can stabilize the linear model (1) in closed-loop using the control law (5)-(6).

5. **Root-Locus Design.**
   (a) (3 points) Sketch the root-locus diagram of $L(s)$. Is it possible to select a gain such that the proportional control law (6) stabilizes the closed-loop? Explain.
   (b) (2 points) Describe how could you select a gain on the root-locus diagram to set the closed-loop natural frequency to a desired $\omega_n > 0$.
   (c) (1 point) Can you arbitrarily set both the closed-loop natural frequency, $\omega_n > 0$, *and* the damping ratio, ζ? Explain using the root-locus diagram from (a).
   (d) (6 points) Introduce a derivative term (a zero) in the controller, modify $L(s)$ accordingly, and repeat parts (a)–(c). *You do not need to introduce an extra pole!*

6. **Frequency Domain Design.**
   (a) (2 points) Sketch the magnitude and phase of the Bode plot of $L(j\omega)$.
   (b) (2 points) Sketch the Nyquist plot of $L(j\omega)$.
   (c) (2 points) Explain how can you use the plot from part (b) to design a proportional control law (6) that stabilizes the closed-loop.
   (d) (6 points) Introduce a derivative term (a zero) in the controller, modify $L(s)$ accordingly, and repeat parts (a)–(c). *You do not need to introduce an extra pole!*

7. **Integral Control.**
   (a) (3 points) Explain using the root-locus diagram why the integral controller:

   $$K(s) = \frac{K}{s}, \qquad K > 0,$$

   cannot stabilize model (1) in closed-loop.
   (b) (1 point (bonus)) Repeat part (a) using a Nyquist plot instead of the root-locus.
   (c) (3 points) Explain using the root-locus diagram how the PI controller:

   $$K(s) = \frac{K(s+z)}{s}, \qquad K > 0,$$

   can stabilize model (1) in closed-loop with z positive and real.
   (d) (1 point (bonus)) Repeat part (c) using a Nyquist plot instead of the root-locus.
   (e) (2 points) What type of target can the controller from part (c) track asymptotically?
   (f) (2 points) What type of torque disturbance can the controller from part (c) reject asymptotically?